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Concept based notes

Maths

Class -XII

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Preface

I am glad to present this book, especially designed to serve the needs of the students. The book has been written keeping in mind the general weakness in understanding the fundamental concepts of the topics. The book is self-explanatory and adopts the "Teach Yourself" style. It is based on question-answer pattern. The language of book is quite easy and understandable based on scientific approach.

Any further improvement in the contents of the book by making corrections, omission and inclusion is keen to be achieved based on suggestions from the readers for which the author shall be obliged.

I acknowledge special thanks to Mr. Rajeev Biyani, *Chairman* & Dr. Sanjay Biyani, *Director* (*Acad.*) Biyani Group of Colleges, who are the backbones and main concept provider and also have been constant source of motivation throughout this Endeavour. They played an active role in coordinating the various stages of this Endeavour and spearheaded the publishing work.

I look forward to receiving valuable suggestions from professors of various educational institutions, other faculty members and students for improvement of the quality of the book. The reader may feel free to send in their comments and suggestions to the under mentioned address.

Author

Unit 1

Chapter – 1

Relations and functions

Q.1 Let A be set of first ten natural numbers. If R be a relation on A defined by $xRy \Leftrightarrow x + 2y = 10$ then

i. Express R and R^{-1} as set of ordered pairs

ii. Find domain of R and R^{-1}

iii. Find range of R and R^{-1}

Ans. Here

A = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10 }

Relation R is defined as

$$xRy \Rightarrow x + 2y = 10$$

$$\Leftrightarrow y = \left(\frac{10 - x}{2}\right)$$

Now for x = 1, $y = \frac{10-1}{2} = \frac{9}{2} \notin A$

Hence 1 is not related to any element of A.

Similarly use can observe that 3, 5, 7, 9 and 10 are also not related to any element of A.

Again use observe that

When $x = 2, y = \frac{10-2}{2} = 4 \in A \Rightarrow 2R4$

When $x = 4, y = \frac{10-4}{2} = 3 \in A \Rightarrow 4R3$

When $x = 6, y = \frac{10-6}{2} = 2 \in A \Rightarrow 6R2$

When $x = 8, y = \frac{10-8}{2} = 1 \in A \Rightarrow 8R1$

Hence

i. R = {(2,4), (4,3), (6,2), (8,1) }

and $R^{-1} = \{(4,2),(3,4),(2,6),(1,8)\}$

ii. Domain of R = {2, 4, 6, 8}

Domain of $R^{-1} = \{4,3,2,1\}$

iii. Range of R = {4, 3, 2, 1}

Range of $R^{-1} = \{2,4,6,8\}$

Q.2 Prove that the relation R on the set N x N defined by (a, b) R (c, d) $\Leftrightarrow$ a+d =b+ c for all (a, b), (c, d) $\in$ N x N is an equivalence relation.

Ans. To prove that the given relation is an equivalence relation we have relation to show that it is reflexive, symmetric and transitive.

1) **Reflexive** – Let (a, b) be an arbitary element of N x N. Then,

(a, b) $\in$ N x N $\Rightarrow$ a+ b = b+ a [by commutativity of addition on N]

$\Rightarrow$ (a, b) R (a, b)

Thus (a, b) R (a, b) for all (a, b) $\in$ N x N

Hence the given relation R is reflexive relation on N x N.

2) **Symmetric** – Let (a. b), (c, d) $\in$ N x N, Such that (a, b) R (c, d)

Since (a, b) R (c, d) $\Rightarrow$ a + d = b + c

$\Rightarrow$ c + b = d + a [by commutativity of addition on N]

$\Rightarrow$ (c, d) R (a, b)

Thus (a, b) R (c, d) $\Rightarrow$ (c, d) R (a, b) for all (a, b), (c, d) $\in$ N x N .

So R is symmetric relation on N x N

3) **Transitive** – Let (a, b), (c, d) and (e, f) $\in$ N x N.

Such that (a, b) R (c, d) and (c, d) R (e, f)

Since (a, b) R (c, d) $\Rightarrow$ a + d = b + c …………………(1)

And (c, d) R (e, f) $\Rightarrow$ c + f = d + e ……………………(2)

Adding equation (1) & (2), we get

$$a + \not{d} + \not{c} + f = b + \not{c} + \not{d} + e$$

$\Rightarrow$ a + f = b+ e

$\Rightarrow$ (a, b) R (e, f)

Thus (a, b) R (c, d) and (c, d) R (e, f) $\Rightarrow$ (a, b) R (e, f) for all (a, b), (c d), (e, f) $\in$ N x N.

So, R is transitive relation on N x N

Hence R being reflexive, symmetric and transitivite is an equivalence relation on N x N.

Hence proved.

Q. 3 On the set N of natural numbers a relation R is defined as

a R b ⇔ a^2-4ab+$3b^2$=0 ∀ (a, b ∈ N) . Prove that R is reflexive but not symmetric not transitivity.

Ans. Given set is N = {1, 2, 3,……..}

Relation defined on N is

a R b ⇔ **a^2-4ab+$3b^2$=0 ∀ a, b ∈ N.**

1) Reflexivity – Let a ∈ N

$$\Leftrightarrow a^2 - 4a.a + 3a^2$$
$$= a^2 - 4a^2 + 3a^2$$
$$= 4a^2 - 4a^2$$
$$= 0$$

⇒ (a, a) ∈R ∀ a ∈ N.

∴ R is reflexive

2) Symmetry - Let a, b ∈ N such that (a, b) ∈ R

$$\therefore (a,b) \in R \Rightarrow a^2 - 4ab + 3b^2 = 0$$
$$\Rightarrow b^2 - 4ba + 3a^2 \neq 0$$
$$\Rightarrow (b,a) \notin R$$

Hence (a, b) R but (b, a) ∉R

∴ R is not symmetric relation

Ex. (3, 1) ∈ R because $3^2 - 4\times 3\times 1 + 3(1)^2$ = 9-12+3 =12-12 = 0

But (1, 3) ∉R because = $(1)^2 - 4(1)(3) + 3(3)^2 = 1 - 12 + 27 \neq 0$

3) Transitivity – Let a, b, c ∈ N such that

(a, b) ∈R and (b, c) ∈R

$$\therefore (a,b) \in R \Rightarrow a^2 - 4ab + 3b^2 = 0$$

and (b, c) ∈ R ⇒ b^2-4bc+$3c^2$

Then it is not necessary true that a^2-4ac+$3c^2$=0

Ex. – (9, 3) ∈because $9^2 - 4(9)(3) + 3(3)^2$ = 81 – 108 + 27 = 0

and (3, 1) ∈ R because $3^2 - 4(3)(1) + 3(1)^2$ = 9-12+3 = 0

but (9, 1) ∉ R because $9^2 - 4(9)(1) + 3(1)^2$ = 81-36+3 ≠ 0

R is not transitivity. Hence from (1), (2) & (3) it is clear that R is reflexive but not symmetric and transitivity.

Q. 4 Let A = {1,2,3} then give examples of relations which are –

1) Reflexive, symmetric and transitive

2) Symmetric and transitive but not reflexive

3) Reflexive and transitive but not symmetric

4) Reflexive and symmetric but not transitive

Ans. 1) $R_1 = \{(1,1),(2,2),(3,3)\}$ is reflexive, symmetric and transitive

2) $R_2 = \{(1,1),(2,2)\}$ is symmetric and transitive but not reflexive

3) $R_3 = \{(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)\}$ is reflexive and transitive but not symmetric

4) $R_4 = \{(1,1),(1,2),(2,1),(2,2),(2,3),(3,2),(3,3)\}$ is reflexive and symmetric but not transitive

Q. 5 If a, b, ∈ {1, 2, 3, 4}, then check whether the following is function or not

f = {(a, b):b=a+1} also find its range.

Ans. Here f = { (1, 2), (2, 3), (3, 4)}. Here we observe that an element 4 of the given set is not related to any element of the given set. So f is not a function.

Q. 6 If $f(x) = \left(\frac{x-3}{x+1}\right)$ then find $f[f\{f(x)\}]$

Ans. f (x) = $\frac{x-3}{x+1}$

Now $f\{f(x)\} = \frac{f(x)-3}{f(x)+1} = \frac{\left(\frac{x-3}{x+1}\right)-3}{\left(\frac{x-3}{x+1}\right)+1}$ = $= \frac{x-3-3x-3}{x-3+x+1} = \frac{-2x-6}{2x-2} = \left(\frac{x+3}{1-x}\right)$

Again –

f [f {f(x)}] = f $\left[\frac{x+3}{1-x}\right]$ = $\frac{\left(\frac{x+3}{1-x}\right)-3}{\left(\frac{x+3}{1-x}\right)+1}$

$=\frac{x+3-3+3x}{x+3+1-x}=\frac{4x}{4}=x$

$\therefore f[f\{f(x)\}]=x$

Q. 7 Prove that the function f: R $\rightarrow$ R, f(x) = cos x is many one function?

Ans. Given f: R $\rightarrow$ R such that f (x) = cos x

many one function: -

Let a, b $\in$ R such that f (a) = f (b)

$\Rightarrow \cos a = \cos b$

$\Rightarrow a = 2n\pi \pm b, n \in I$

$\therefore$ f is many one function

Into function – Let y $\in$ R (Co – domain)

If it is possible let f (x) =y

$\Rightarrow \cos x = y$

$\Rightarrow x = \cos^{-1} y$

x will exist if $-1 \le y \le 1$

When $y \in R-[-1,1]$ then pre – image of y does not exist in R (Domain)

Hence f is not on to function

$\therefore$ f is in to function

Hence f is many – one in to function.

Q. 8 If f: R $\rightarrow$ R, f (x) = 2x – 3 and g: R $\rightarrow$ R, g(x) = $\left(\frac{x+3}{2}\right)$ then prove that fog = gof = I_R

Ans. Given functions $f : R \rightarrow R$, f (x) = 2x-3

$$g : R \rightarrow R, g(x) = \frac{x+3}{2}$$

$gof : R \rightarrow R$, so

(gof) (x) = g [f(x)] = g (2x-3) = $\frac{(2x-3)+3}{2} = \frac{2x}{2} = x$

gof (x) = x .... (1)

Again

fog (x) = f [g(x)] =

$= f\left[\frac{x+3}{2}\right] = 2\left(\frac{x+3}{2}\right) - 3$

$= x + 3 - 3 = x$

fog (x) = x .... (2)

and I_R : R $\rightarrow$ R such that $I_R(x) = x, \forall x \in R$ ....(3)

from (1), (2) & (3) we get,

fog = gof = I_R

Hence proved

Q. 9 Let f: R $\rightarrow$ R be defined by f (x) = 3x – 7. Show that f is invertible and hence find f^{-1}

Ans. A function f is invertible if f is a bijection

1) **Injectivity –** Let x, y $\in$ R then

f(x) = f (y)

$\Rightarrow 3x - 7 = 3y - 7$

$\Rightarrow x = y$

Thus f (x) = f(y) $\Rightarrow$ x=y for all x, y $\in$ R. So, f is an injection

2) **Surjectivity –** Let y be an arbitary element of R, then f (x) = y

$\Rightarrow 3x - 7 = y$

$\Rightarrow x = \frac{y+7}{3}$

Clearly $\left(\frac{y+7}{3}\right) \in R$ for all y $\in$ R

Thus for all y $\in$ R, there exists $x = \frac{y+7}{3} \in R$ such that

$$f(x) = f\left(\frac{y+7}{3}\right)$$

$$= 3\left(\frac{y+7}{3}\right) - 7$$

$$f(x) = y$$

$\therefore$ f is surjection. Hence $f : R \to R$ is bijection. Consequently it is invertible

Let f(x) = y

$$\Rightarrow 3x - 7 = y$$

$$\Rightarrow x = \left(\frac{y+7}{3}\right)$$

$$\Rightarrow f^{-1}(y) = \left(\frac{y+7}{3}\right)$$

Therefore, $f^{-1} : R \to R$ is given by

$$f^{-1}(x) = \left(\frac{x+7}{3}\right)$$

Chapter – 2

Binary Operation

Q. 1. Discuss the commutativity and associativity of the binary operation $*$ on R defined by

a * b = $\frac{ab}{4}$ for all a, b $\in$ R.

Ans. Commutativity –

a * b = $\frac{ab}{4}$ and b*a = $\frac{ba}{4}$

We know that multiplication on R is commutative

$\therefore \frac{ab}{4} = \frac{ba}{4}$ for all a, b $\in$ R

$\Rightarrow a * b = b * a$ for all a, b $\in$ **R**

So '*' is commutative on R.

Associatively –

Let a, b, c $\in$ R then

(a * b) * c $= \dfrac{\left(\frac{ab}{4}\right)(c)}{4} = \dfrac{abc}{16}$ ....(1)

and a * (b * c) = a * $\left(\frac{bc}{4}\right) = \dfrac{a\left(\frac{bc}{4}\right)}{4} = \dfrac{abc}{16}$ ……………………..(2)

From (1) & (2), we observe that a * (b*c) = (a*b) *c

Hence '*' is associative.

Q. 2. Let '*' be a binary operation on set Q – {1} defined by a*b = a+ b – a b , a, b $\in$ Q - {1}

Find the identity element with respect to * on Q. Also prove that every element of Q – {1} is invertible.

Ans. Let the identity element e exist in Q – {1} w.r.t * on Q - {1}, then

a *e = a = e*a for all a $\in$Q - {1}

$\Rightarrow$ a*e = a for all a $\in$ Q - {1}

$\Rightarrow$ a+ e – ae =a

$\Rightarrow$ e (1-a) = 0

$\Rightarrow$ e = 0

Thus o, is the identity element for * on Q – {1}. Let a be an arbitary element of Q – {1} and

let b be inverse of a, then.

a * b = 0 = b *a [0 is identity element]

$\Rightarrow$ a * b = 0

$\Rightarrow$ a + b – ab = 0

b (1 – a) = - a

$\Rightarrow$ b = $\left(\dfrac{a}{a-1}\right)$ $\left[\begin{array}{l} \therefore a \in Q-\{1\} \\ \therefore a-1 \neq 0 \end{array}\right.$

Since $a \in Q-\{1\}$, therefore b= $\left(\dfrac{a}{a-1}\right) \in Q-\{1\}$

Thus every element of Q – {1} is invertible and the inverse of an element a is $\left(\frac{a}{a-1}\right)$

Q. 3 Let * be an associative binary operation on a set S and a be an invertible element of

S then $(a^{-1})^{-1} = a$

Ans. Let e be the identity element in S for the binary operation * on S, then

$a * a^{-1} = e = a^{-1} * a$

$\Rightarrow a^{-1} * a = e = a * a^{-1}$

$\Rightarrow$ a is inverse of a^{-1}

$\Rightarrow$ ***a*** = $(a^{-1})^{-1}$

Hence proved

Chapter – 3

Inverse trigonometric functions

Q. 1 Find the principal values of the followings: -

1) $\cos^{-1}\left(\frac{-1}{2}\right)$ (2) $\sec^{-1}\left(\sqrt{2}\right)$

3) $\operatorname{cosec}^{-1}(1)$ (4) $\cot^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

Ans. (1) Let $\cos^{-1}\left(\frac{-1}{2}\right) = \theta$

$\Rightarrow \cos\theta = -1/2$

$$\Rightarrow \cos\theta = -\cos\left(\frac{\pi}{3}\right)$$

$$\Rightarrow \cos\theta = \cos\left(\pi - \frac{\pi}{3}\right)$$

$$\Rightarrow \cos\theta = \cos\left(\frac{2\pi}{3}\right)$$

$$\Rightarrow \theta = \left(\frac{2\pi}{3}\right)$$

$$\Rightarrow \cos^{-1}\left(\frac{-1}{2}\right) = \frac{2\pi}{3}$$

Hence principal value of $\cos^{-1}\left(\frac{-1}{2}\right)$ is $\left(\frac{2\pi}{3}\right)$

(2) Let $\sec^{-1}(-\sqrt{2}) = \theta$

$$\sec\theta = -\sqrt{2}$$

$$\Rightarrow \sec\theta = -\sec\left(\frac{\pi}{4}\right)$$

$$\Rightarrow \sec\theta = \sec\left(\pi - \frac{\pi}{4}\right)$$

$$\Rightarrow \sec\theta = \sec\left(\frac{3\pi}{4}\right)$$

$$\Rightarrow \theta = \left(\frac{3\pi}{4}\right)$$

$$\Rightarrow \sec^{-1}(-\sqrt{2}) = \left(\frac{3\pi}{4}\right)$$

Hence principal value of $\sec^{-1}(-\sqrt{2})$ is $\left(\frac{3\pi}{4}\right)$

3) Let $\text{cosec}^{-1}(+1) = \theta$

$$\Rightarrow \text{cosec}\theta = +1$$

$$\Rightarrow \text{cosec}\theta = \text{cosec}\,\pi/2$$

$$\Rightarrow \theta = \pi/2$$

$$\Rightarrow \text{cosec}^{-1}(1) = \pi/2$$

Hence principal value of $\text{cosec}^{-1}(1)$ is $\pi/2$

4) Let $\cot^{-1}\left(\frac{-1}{\sqrt{3}}\right) = \theta$

$\Rightarrow \cot\theta = -\frac{1}{\sqrt{3}}$

$\Rightarrow \cot\theta = -\cot\left(\frac{\pi}{3}\right)$

$\Rightarrow \cot\theta = \cot(\pi - \pi/3)$

$\Rightarrow \cot\theta = \cot\left(\frac{2\pi}{3}\right)$

$\Rightarrow \theta = \left(\frac{2\pi}{3}\right)$

$\Rightarrow \cot^{-1}\left(\frac{-1}{\sqrt{3}}\right) = \left(\frac{2\pi}{3}\right)$

Hence principal value of $\cot^{-1}\left(\frac{-1}{\sqrt{3}}\right) is \left(\frac{2\pi}{3}\right)$.

Q. 2 Prove that $\tan\left[\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right)\right]+\tan^{-1}\left[\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\frac{a}{b}\right]=\frac{2b}{a}$

Ans. Let $\frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right)=\theta$

$\Rightarrow \cos 2\theta = \left(\frac{a}{b}\right)$

LHS = $\tan\left(\frac{\pi}{4}+\theta\right)+\tan\left(\frac{\pi}{4}-\theta\right)$

$$= \frac{\tan\frac{\pi}{4} + \tan\theta}{1 - \tan\frac{\pi}{4} \times \tan\theta} + \frac{\tan\left(\frac{\pi}{4}\right) - \tan\theta}{1 + \tan\left(\frac{\pi}{4}\right)\tan\theta}$$

$$= \frac{1 + \tan\theta}{1 - \tan\theta} + \frac{1 - \tan\theta}{1 + \tan\theta}\left[\tan\frac{\pi}{4} = 1\right]$$

$$= \frac{(1 + \tan\theta)^2 + (1 - \tan\theta)^2}{(1 - \tan\theta)(1 + \tan\theta)}$$

$$= \frac{1 + \tan^2\theta + 2\tan\theta + 1 + \tan^2\theta - 2\tan\theta}{(1 - \tan^2\theta)}$$

$$= 2\left(\frac{1 + \tan^2\theta}{1 - \tan^2\theta}\right)$$

$$= \frac{2}{\frac{1 - \tan^2\theta}{1 + \tan^2\theta}} = \frac{2}{\cos^2\theta}$$

$$= \frac{2b}{a} = \text{RHS}$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Hence proved.

Q. 3 If $\sin^{-1}\left(\frac{5}{x}\right) + \sin^{-1}\left(\frac{12}{x}\right) = 90^\circ$ **then find the value of x**

Ans. $\therefore \sin^{-1}\left(\frac{5}{x}\right) + \sin^{-1}\left(\frac{12}{x}\right) = 90^\circ$

$$\Rightarrow \sin^{-1}\left(\frac{5}{x}\right) = 90^{\circ} - \sin^{-1}\left(\frac{12}{x}\right)$$

$$\Rightarrow \sin^{-1}\left(\frac{5}{x}\right) = \cos^{-1}\left(\frac{12}{x}\right)$$

$$\Rightarrow \left[\sin^{-1}x + \cos^{-1}x = 90^{\circ}\right]$$

$$\Rightarrow \cos^{-1}\left(\sqrt{1 - \frac{25}{x^2}}\right) = \cos^{-1}\left(\frac{12}{x}\right)$$

$$\Rightarrow \sqrt{1 - \frac{25}{x^2}} = \frac{12}{x}$$

Squaring both sides, we get

$$1 - \frac{25}{x^2} = \frac{144}{x^2}$$

$$\Rightarrow \frac{x^2 - 25}{x^2} = \frac{144}{x^2}$$

$$\Rightarrow x^2 - 25 = 144$$

$$\Rightarrow x^2 = 169$$

$$\Rightarrow x = \pm 13$$

Since x = - 13 does not satisfy given equation . So x = 13 is correct solution.

Q. 4 Prove that

$$\cos\left[\tan^{-1}\{\sin(\cot^{-1} x)\}\right] = \sqrt{\frac{x^2+1}{x^2+2}}$$

Ans. We have

$$\sin(\cot^{-1}(x)) = \sin\left(\sin^{-1}\frac{1}{\sqrt{1+x^2}}\right)$$

$$= \frac{1}{\sqrt{1+x^2}}$$

$$\therefore \cos\left\{\tan^{-1}\frac{1}{\sqrt{1+x^2}}\right\} = \cos\left\{\cos^{-1}\frac{\sqrt{1+x^2}}{\sqrt{2+x^2}}\right\}$$

$$= \frac{\sqrt{1+x^2}}{\sqrt{2+x^2}} = \sqrt{\frac{1+x^2}{2+x^2}}$$

Hence proved

Q. 5 If $\cos^{-1}\left(\frac{x}{a}\right)+\cos^{-1}\left(\frac{y}{b}\right)=\alpha$ **then prove that**

$$\frac{x^2}{a^2}-\frac{2xy}{ab}\cos\alpha+\frac{y^2}{b^2}=\sin^2\alpha$$

Ans. Given $\cos^{-1}\left(\frac{x}{a}\right)+\cos^{-1}\left(\frac{y}{b}\right)=\alpha$

$$=\cos^{-1}\left\{\frac{x}{a}\times\frac{y}{b}-\sqrt{1-\frac{x^2}{a^2}}\times\sqrt{1-\frac{y^2}{b^2}}\right\}=\alpha$$

$$\left[\cos^{-1}x+\cos^{-1}y=\cos^{-1}\{\ xy+\sqrt{1-x^2}\sqrt{1-y^2}\ \}\right]$$

$$\Rightarrow\frac{xy}{ab}-\sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}=\cos\alpha$$

$$\Rightarrow\left(\frac{xy}{ab}-\cos\alpha\right)^2=\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{b^2}\right)$$

$$\Rightarrow\frac{x^2y^2}{a^2b^2}-\frac{2xy}{ab}\cos\alpha+\cos^2\alpha=1-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{x^2y^2}{a^2b^2}$$

$$\Rightarrow\frac{x^2}{a^2}-\frac{2xy}{ab}\cos\alpha+\frac{y^2}{b^2}=1-\cos^2\alpha$$

$$\Rightarrow\frac{x^2}{a^2}-\frac{2xy}{ab}\cos\alpha+\frac{y^2}{b^2}=\sin^2\alpha$$

Hence proved

Q. 6 Solve the following equation –

$$\tan^{-1}\left(\frac{1}{1+2x}\right)+\tan^{-1}\left(\frac{1}{4x+1}\right)=\tan^{-1}\left(\frac{2}{x^2}\right)$$

Ans. Given $\tan^{-1}\left(\frac{1}{1+2x}\right)+\tan^{-1}\left(\frac{1}{4x+1}\right)=\tan^{-1}\left(\frac{2}{x^2}\right)$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{1}{1+2x}+\frac{1}{1+4x}}{1-\left(\frac{1}{1+2x}\right)\left(\frac{1}{1+4x}\right)}\right]=\tan^{-2}\left(\frac{2}{x^2}\right)$$

$$\Rightarrow\left[\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$\Rightarrow \frac{4x+1+1+2x}{(1+2x)(4x+1)-1}=\frac{2}{x^2}$$

$$\Rightarrow \frac{6x+2}{(\not{1}+4x+8x^2+2x-\not{1})}=\frac{2}{x^2}$$

$$\Rightarrow \frac{\not{2}(3x+1)}{\not{2}(4x^2+3x)}=\frac{2}{x^2}$$

$$\Rightarrow 3x^3+x^2=8x^2+6x$$

$$\Rightarrow 3x^3-7x^2-6x=0$$

$$\Rightarrow x\left[3x^2-7x-6\right]=0$$

$$\Rightarrow x\left[3x^2-9x+2x-6\right]=0$$

$$\Rightarrow x\left[3x(x-3)+2(x-3)\right]=0$$

$$\Rightarrow x\left[(3x+2)(x-3)\right]=0$$

$$\Rightarrow x=0,3,\frac{-2}{3}$$

Practice Problems –

1) Find principal values of the followings

i) $\sin^{-1}\left(-\frac{1}{2}\right)$ ii) $\tan^{-1}\left(-\sqrt{3}\right)$

2) Solve the following equation

$$\sec^{-1}\left(\frac{x}{a}\right)-\sec^{-1}\left(\frac{x}{b}\right)=\sec^{-1}(b)-\sec^{-1}(a)$$

3) If $\tan^{-1}(3x)+\tan^{-1}(2x)=\frac{\pi}{4}$ then find value of x.

4) Solve the following - $\tan^{-1}(x-1)+\tan^{-1}(x)+\tan^{-1}(x+1)=\tan^{-1}(3x)$.

Chapter – 2

Determinant

Q. 1 Find determinant of A = $\begin{bmatrix} -1 & 6 & -2 \\ 2 & 1 & 1 \\ 4 & 1 & -3 \end{bmatrix}$

Ans. |A| = (-1) $\begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} - 6 \begin{vmatrix} 2 & 1 \\ 4 & -3 \end{vmatrix} - 2 \begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix}$

= -1 (-3-1) -6 (-6-4) -2 (2-4) = 4+60+4

|A| = 68

Q. 2 Find determinant of A = $\begin{vmatrix} 1 & 2 & -1 & 3 \\ 2 & 1 & -2 & 3 \\ 3 & 1 & 2 & 1 \\ 1 & -1 & 0 & 2 \end{vmatrix}$

Ans. |A| = $1\begin{vmatrix} 1 & -2 & 3 \\ 1 & 2 & 1 \\ -1 & 0 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & -2 & 3 \\ 3 & 2 & 1 \\ 1 & 0 & 2 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 & 3 \\ 3 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 1 & -2 \\ 3 & 1 & 2 \\ 1 & -1 & 0 \end{vmatrix}$

= 1{1(4-0) +2(2+1) +3(0+2) {-2{2(4-0) +2(6-1) +3(0-2)}- 1{2(2+1)-1(6-1)

+3(-3-1)}-3{2(0+2)-1(0-2)-2(-3-1)}

= 1{4+6+6}-2{8+10-6}-1{6-5-12}-3{4+2+8}

= 1{16}-2{12}-1{-11}-3{14}

= -39

Q. 3 Check whether the following matrix is singular or not

A = $\begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$

Ans. A matrix is singular if $|A| = 0$

$$|A| = \begin{vmatrix} 1 & 0 & 2 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{vmatrix} = 1[12-12]-0[6-6]+2[4-4]=0$$

$$\Rightarrow |A| = 0$$

Hence A is singular matrix.

Q. 4 Find the mines and cofactors of elements of the determinant $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$

Ans. We have

$$M_{11} = \begin{vmatrix} 0 & 4 \\ 5 & -7 \end{vmatrix} = 0 - 20 = -20$$

$$M_{12} = \begin{vmatrix} 6 & 4 \\ 1 & -7 \end{vmatrix} = -42 - 4 = -46$$

$$M_{13} = \begin{vmatrix} 6 & 0 \\ 1 & 5 \end{vmatrix} = 30 - 0 = 30$$

$$M_{21} = \begin{vmatrix} -3 & 5 \\ 5 & -7 \end{vmatrix} = 21 - 25 = -4$$

$$M_{22} = \begin{vmatrix} 2 & 5 \\ 1 & -7 \end{vmatrix} = -14 - 5 = -19$$

$$M_{23} = \begin{vmatrix} 2 & -3 \\ 1 & 5 \end{vmatrix} = 10 + 3 = 13$$

$$M_{31} = \begin{vmatrix} -3 & 5 \\ 0 & 4 \end{vmatrix} = -12 - 0 = -12$$

$$M_{32} = \begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix} = 8 - 30 = -22$$

$$M_{33} = \begin{vmatrix} 2 & -3 \\ 6 & 0 \end{vmatrix} = 0 + 18 = 18$$

Co- factors –

$$A_{11} = (-1)^{1+1}.M_{11} = (-1)^2 \times (-20) = -20$$

$$A_{12} = (-1)^{1+2}.M_{12} = (-1)^3 \times (-46) = 46$$

$$A_{13} = (-1)^{1+3}.M_{13} = (-1)^4 \times 30 = 30$$

$$A_{21} = (-1)^{2+1}.M_{21} = (-1)^3 \times (-4) = 4$$

$$A_{22} = (-1)^{2+2}.M_{22} = (-1)^4 \times (-19) = -19$$

$$A_{23} = (-1)^{2+3}.M_{23} = (-1)^5 \times (13) = -13$$

$$A_{31} = (-1)^{3+1}.M_{31} = (-1)^4 \times (-12) = -12$$

$$A_{32} = (-1)^{3+2}.M_{32} = (-1)^5 \times (-22) = 22$$

$$A_{33} = (-1)^{3+3}.M_{33} = (-1)^6 \times 18 = 18$$

Q. 5 If W is one of the imaginary cubs root of unity, find the value of

$$\Delta = \begin{vmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix}$$

Ans. Given $\Delta = \begin{vmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix}$

$C_1 \to C_1 + C_2 + C_3$ gives

$$\Delta = \begin{vmatrix} 1+w+w^2 & w & w^2 \\ w+w^2+1 & w^2 & 1 \\ w^2+w+1 & 1 & w \end{vmatrix}$$

Since $1 + w + w^2 = 0$

So $\Delta = \begin{vmatrix} 0 & w & w^2 \\ 0 & w^2 & 1 \\ 0 & 1 & w \end{vmatrix}$

Now finding its determinant & expending along first column

$$\Delta = 0\left[w^3 - 1\right] - 0\left[w^2 - w^2\right] + 0\left[w - w^4\right] = 0$$

Q. 6 Prove that $\begin{vmatrix} a & b & ax+by \\ b & c & bx+cy \\ ax+by & bx+cy & 0 \end{vmatrix} = (b^2-ac)\left(ax^2+2bxy+cy^2\right)$

Ans. Let A = $\begin{vmatrix} a & b & ax+by \\ b & c & bx+cy \\ ax+by & bx+cy & 0 \end{vmatrix}$

Appling $c_3 \to c_3 - xc_1 - yc_2$, we get

A = $\begin{vmatrix} a & b & ax+by-ax-by \\ b & c & bx+cy-bx-cy \\ ax+by & bx+cy & 0-x(ax+by)-y(bx+cy) \end{vmatrix}$

A = $\begin{vmatrix} a & b & 0 \\ b & c & 0 \\ ax+by & bx+cy & -(ax^2+2bxy+cy^2) \end{vmatrix}$

Now expending along c_3 we get

$$A = 0\left[b\left(bx+cy\right)-c(ax+by)\right]-0[a(bx+cy)-a(ax+by)]-(ax^2+2bxy+cy^2)[ac-b^2]$$

$$A = -(ax^2+2bxy+cy^2)(ac-b^2)$$

$$A = (b^2-ac)(ax^2+2bxy+cy^2)$$

Hence proved

Q. 7 Find the area of triangle with vertices at the points (3, 8), (-4, 2) and (5,-1).

Ans. Let A (3, 8), B (-4, 2), C (5,-1) are three given vertices of triangle. So the area of ΔABC is given by –

$$\square = \frac{1}{2}\begin{vmatrix} 3 & 8 & 1 \\ -4 & 2 & 1 \\ 5 & -1 & 1 \end{vmatrix}$$

$$□ = \frac{1}{2}\left[3\{2+1\} - 8\{-4-5\} + 1\{4-10\}\right]$$

$$□ = \frac{1}{2}\left[3\times 3 + 8\times 9 + 1\times(-6)\right]$$

$$□ = \frac{1}{2}\left[9 + 72 - 6\right]$$

$$□ = \frac{75}{2}$$

Q. 8 If the points (a, 0), (0, b) and (1, 1) are collinear, Prove that a + b = ab

Ans. The given points are collinear so

$$\begin{vmatrix} a & 0 & 1 \\ 0 & b & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow a[b-1] + 1[0-b] = 0$$

$$\Rightarrow ab - a - b = 0$$

$$\Rightarrow ab = a + b$$

Hence proved.

Q. 9 using determinant find the equation of the line joining the points (1, 2) and (3, 6).

Ans. Let P (x, y) be a point on line AB i.e. the points A (1, 2), B (3, 6) and P(x, y) are collinear so

$$□ ABP = 0$$

$$\Rightarrow \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 3 & 6 & 1 \\ x & y & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & 2 & 1 \\ 3 & 6 & 1 \\ x & y & 1 \end{vmatrix} = 0$$

$$\Rightarrow 1[6-y] - 2[3-x] + 1[3y-6x] = 0$$

$$\Rightarrow \cancel{6} - y - \cancel{6} + 2x + 3y - 6x = 0$$

$$\Rightarrow -4x + 2y = 0$$

$$\Rightarrow y = 2x$$

Which is the required equation of line.

Q. 10 Find the adjoint of the matrix

$$A = \begin{vmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{vmatrix}$$

Ans. First we have to find cofactors

$$c_{11} = (-1)^{1+1} \times 7 = 7$$

$$c_{12} = (-1)^{1+2} \times (1) = -1$$

$$c_{13} = (-1)^{1+3} \times (-1) = -1$$

$$c_{21} = (-1)^{2+1} \times (3) = -3$$

$$c_{22} = (-1)^{2+2} \times (1) = 1$$

$$c_{23} = (-1)^{2+3} \times (0) = 0$$

$$c_{31} = (-1)^{3+1} \times (-3) = -3$$

$$c_{32} = (-1)^{3+2} \times (0) = 0$$

$$c_{33} = (-1)^{3+3} \times (1) = 1$$

$$\therefore \text{adj}A = \begin{vmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 0 \end{vmatrix}^T$$

$$\therefore \text{adj}A = \begin{vmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

Q. 11 Find the inverse of matrix $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$.

Ans. $|A| = 8 + 3 = 11 \neq 0$

So, A is non – singular matrix and therefore it is invertible. Now finding co-factors

$$c_{11} = (-1)^{1+1} \times 4 = 4$$

$$c_{12} = (-1)^{1+2} \times 3 = -3$$

$$c_{21} = (-1)^{2+1} \times (-1) = 1$$

$$c_{22} = (-1)^{2+2} \times 2 = 2$$

$$\therefore \text{adj } A = \begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix}^T$$

$$\text{adj}A = \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$$

Hence $A^{-1} = \frac{1}{|A|}$.adj(A)

$$= \frac{1}{11}\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 4/11 & 1/11 \\ -3/11 & 2/11 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} 4/11 & 1/11 \\ -3/11 & 2/11 \end{bmatrix}$$

Q. 12 Find the inverse of $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ **and verify that** $A^{-1}A = I_3$

Ans. From question No. 10 we find that

adj A = $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

$$|A| = 1[16-9]-3[4-3]+3[4-3] = 7-3-3 = 1 \neq 0$$

So A is invertible

Hence

$$A^{-1} = \frac{1}{|A|}\text{adj}A$$

$$A^{-1} = \frac{1}{1}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Now

$$A^{-1}A = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} 7-3-3 & 21-12-9 & 21-9-12 \\ -1+1+0 & -3+4+0 & -3+3+0 \\ -1+0+1 & -3+0+3 & -3+0+4 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore A^{-1}A = I_3$$

Hence verified

Q. 13 Solve the following system of equations by using cramer's rule.

$$x + 2y = 3$$

$$4x + 8y = 12$$

Ans. We have

$$D = \begin{vmatrix} 1 & 2 \\ 4 & 8 \end{vmatrix} = 8 - 8 = 0$$

$$D_1 = \begin{vmatrix} 3 & 2 \\ 12 & 8 \end{vmatrix} = 24 - 24 = 0$$

$$D_2 = \begin{vmatrix} 1 & 3 \\ 4 & 12 \end{vmatrix} = 12 - 12 = 0$$

Since D, D_1 and D_2 all are equal to zero so the given system of equations has infinitely many solutions.

Let y=k then from equations x+2y=3

x+2k=3

x= 3-2k

Hence, x=3-2k, y= k is the solution of the given system of equations, where k is arbitrary real number.

Q. 14 solve the following system of equations by cramer's rule

x – 2y = 4

- 3x +5y = -7

Ans. We have

$$D = \begin{vmatrix} 1 & -2 \\ -3 & 5 \end{vmatrix} = 5 - 6 = -1 \neq 0$$

$$D_1 = \begin{vmatrix} 4 & -2 \\ -7 & 5 \end{vmatrix} = 20 - 14 = 6$$

$$D_2 = \begin{vmatrix} 1 & 4 \\ -3 & -7 \end{vmatrix} = -7 + 12 = 5$$

So, by Cramer's rule, we have

$$x = \frac{D_1}{D} = \frac{6}{-1} = -6$$

$$y = \frac{D_2}{D} = \frac{5}{-1} = -5$$

x= -6, y= -5 is required solution.

Q. 15 Solve the following system of equations

2x+3y+4z=0

x+ y+ z = 0

2x-y+3z=0

Ans. We have

$$D = \begin{vmatrix} 2 & 3 & 4 \\ 1 & 1 & 1 \\ 2 & -1 & 3 \end{vmatrix}$$
$$= 2(3+1) - 3(3-2) + 4(-1-2)$$
$$= 8 - 3 - 12$$
$$D = -7 \neq 0$$

So, the given system of equations has only the trivial solutions i.e x=0, y=0, z=0

Q. 16 Solve the following homogeneous system of equations

x+y-2z=0..(1)
2x+y-3z=0……………………………………..(2)
5x+4y-9z=0……………………………………(3)

Ans. We have,

$$D = \begin{vmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{vmatrix}$$
$$= 1(-9+12) - 1(-18+15) - 2(8-5)$$
$$= 3 + 3 - 6$$
$$D = 0$$

So, the system of equations has infinitely many solutions.

Consider eq. (1) & (2). Put z=k in equations(1) and (2), we get –

x+y =2k
2x+y=3k

Solving these equations by cramer's rule

$$D = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1 - 2 = -1$$

$$D_1 = \begin{vmatrix} 2k & 1 \\ 3k & 1 \end{vmatrix} = 2k - 3k = -k$$

$$D_2 = \begin{vmatrix} 1 & 2k \\ 2 & 3k \end{vmatrix} = 3k - 4k = -k$$

$$\therefore x = \frac{D_1}{D} = \frac{-k}{-1} = k$$

$$y = \frac{D_2}{D} = \frac{\not{-}k}{\not{-}1} = k$$

x=k, y=k and z=k gives the solution for each value of k.

Q. 17 Use matrix method to solve the following system of equations –

5x – 7y =2

7x – 5y =3

Ans. The given system of equations can be written as

$$\begin{bmatrix} 5 & -7 \\ 7 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Or A x = B, where

$$A = \begin{bmatrix} 5 & -7 \\ 7 & -5 \end{bmatrix}, \; X = \begin{bmatrix} x \\ y \end{bmatrix}, \; B \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

So, the solution is given by $X = A^{-1}B$. So the find A^{-1} we have to find co factors

$$C_{11} = (-1)^{1+1}(-5) = -5$$

$$C_{12} = (-1)^{1+2}(7) = -7$$

$$C_{21} = (-1)^{2+1}(-7) = 7$$

$$C_{22} = (-1)^{2+2}(5) = 5$$

$$\therefore \text{adj } A = \begin{bmatrix} -5 & -7 \\ 7 & 5 \end{bmatrix}^T = \begin{bmatrix} -5 & 7 \\ -7 & 5 \end{bmatrix}$$

$$|A| = -25 + 49 = 24$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj}(A) = \frac{1}{24} \begin{bmatrix} -5 & 7 \\ -7 & 5 \end{bmatrix}$$

$$X = A^{-1}B$$

$$X = \frac{1}{24}\begin{bmatrix} -5 & 7 \\ -7 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$X = \frac{1}{24}\begin{bmatrix} -10+21 \\ -14+15 \end{bmatrix} = \frac{1}{24}\begin{bmatrix} 11 \\ 1 \end{bmatrix}$$

$$X = \begin{bmatrix} 11/24 \\ 1/24 \end{bmatrix}$$

Hence $x = \frac{11}{24}$ and y= $\frac{1}{24}$

Q. 18 Show that the following system of equations is consistent –

2x – y + 3z = 5
3x + 2y – z = 7
4x + 5y – 5z =9

Ans. The given of equation can be written as –

$$\begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & -1 \\ 4 & 5 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$$

A X =B

Where $A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & -1 \\ 4 & 5 & -5 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$

Now $|A| = \begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & -1 \\ 4 & 5 & -5 \end{vmatrix}$

$= 2[-10+5]+1[-15+4]+3[15-8]$

$= -10-11+21 = 0$

So A is singular. So the given system of equation is either inconsistent on consistent with infinitely many solutions according as (adj A) B $\neq$0 or (adj A) B= 0 respectively.

$c_{11} = (-1)^{1+1}(-10+5) = -5$

$c_{12} = (-1)^{1+2}(-15+4) = 11$

$c_{13} = (-1)^{1+3}(15-8) = 7$

$c_{21} = (-1)^{2+1}(5-15) = 10$

$c_{22} = (-1)^{2+2}(-10-12) = -22$

$c_{23} = (-1)^{2+3} \times (10+4) = -14$

$c_{31} = (-1)^{3+1}(1-6) = -5$

$c_{32} = (-1)^{3+2}(-2-9) = 11$

$c_{33} = (-1)^{3+3}(4+3) = 7$

$$\therefore \text{adj } A = \begin{bmatrix} -5 & 11 & 7 \\ 10 & -22 & -14 \\ -5 & 11 & 7 \end{bmatrix} = \begin{bmatrix} -5 & 10 & -5 \\ 11 & -22 & 11 \\ -7 & -14 & 7 \end{bmatrix}$$

$$\therefore (\text{adj } A)(B) = \begin{bmatrix} -5 & 10 & -5 \\ 11 & -22 & -11 \\ 7 & -14 & 7 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$$

$$= \begin{bmatrix} -25+70-45 \\ 55-154+99 \\ 35-98+63 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore (\text{adj } A)(B) = 0$$

Thus AX=B has infinitely many solutions and the given system of equation is consistent.

Unit – III

Chapter – 1

Continuity and Differentiability

1. Check the continuity of the function f(x) at the origin :

$$f(x) = \begin{cases} \dfrac{|x|}{x}; x \neq 0 \\ 1; x = 0 \end{cases}$$

Ans. We have to show that the given function is continuous at x= 0, so

LHL $\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h)$

$= \lim_{h\to 0} f(-h)$

$= \lim_{h\to 0} \frac{|-h|}{(-h)} = \lim_{h\to o} \frac{h}{-h} = -1$

$\therefore LHL = -1$

RHL $\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h)$

$= \lim_{h\to 0} f(h)$

$= \lim_{h\to 0} \frac{|h|}{(h)} = \lim_{h\to o} \frac{h}{h} = 1$

$\therefore RHL = 1$

Now f(0) = 1

Since LHL ≠ RHL, so the function f(x) is not continuous at the origin.

2. Test the continuity of the function at x= 0

$$f(x) = \begin{cases} \frac{\sin x}{x} + \cos x, & when \quad x \neq 0 \\ 2, & when \quad x = 0 \end{cases}$$

Ans. LHL $\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h)$

$= \lim_{h\to 0} f(-h)$

$= \lim_{h\to 0} \left[\frac{\sin(-h)}{(-h)} + \cos(-h) \right]$

$= \lim_{h\to 0} \frac{\sin(h)}{h} + \lim_{h\to 0} \cos(h)$

$= 1 + 1 = 2$

$\therefore LHL = 2$

RHL $\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h)$

$$= \lim_{h\to 0}\left[\frac{Sin(h)}{(h)} + \cos(h)\right]$$

$$= \lim_{h\to 0}\frac{Sin(h)}{h} + \lim_{h\to 0} Cos(h)$$

$$= 1 + 1 = 2$$

$$\therefore RHL = 2$$

And f(0) = 2

Since f(0) = LHL = RHL

So the given function f(x) is continuous.

3. Find the values of a and b for which the following function is continuous at x = 1.

$$f(x) = \begin{cases} 2x + a & when \quad x > 1 \\ b & when \quad x = 1 \\ 5x - 2 & when \quad x < 1 \end{cases}$$

Ans. Since the function f(x) is continuous at x = 1. So

$$f(1) = f(1-0)$$

$$f(1) = \lim_{h\to o} f(1-h)$$

$$= \lim_{h\to 0} 5(1-h) - 2$$

$$b = \lim_{h\to 0} 5 - 5h - 2$$

$$b = \lim_{h\to 0} 3 - 5h$$

$$\Rightarrow b = 3$$

And f(1+0) = f(1-0)

$$= \lim_{h\to 0} f(1+h) = \lim_{h\to 0} f(1-h)$$

$$= \lim_{h\to o} 2(1+h) + a = \lim_{h\to 0} 5(1-h) - 2$$

$$= 2 + a = 3$$

$$\Rightarrow a = 1$$

Hence a = 1, b = 3

4. Show that the function $f(x) = \begin{cases} \dfrac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}, & x \neq 0 \\ 0 \quad , & x = 0 \end{cases}$ **is discontinuous at x =0**

Ans. LHL $\lim_{x\to 0-} f(x) = \lim_{h\to 0} f(0-h)$

$$= \lim_{h\to 0} f(h) = \lim_{h\to 0} \frac{e^{-1/h}-1}{e^{-1/h}+1}$$

$$= \lim_{h\to 0} \frac{e^{-\frac{1}{1/h}}-1}{e^{-\frac{1}{1/h}}+1}$$

$$= \lim_{h\to 0} \frac{\frac{1}{e^{1/h}}-1}{\frac{1}{e^{1/h}}+1} = \frac{0-1}{0+1} = -1$$

RHL $\lim_{x\to 0+} f(x)$

$$= \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h)$$

$$= \lim_{h\to 0} \frac{e^{\frac{1}{h}}-1}{e^{\frac{1}{h}}+1} = \lim_{h\to 0} \frac{1-\frac{1}{e^{1/h}}}{1+\frac{1}{e^{1/h}}} = \frac{1-0}{1+0} = 1$$

Since LHL ≠ RHL

So that function f(x) is discontinuous at x = 0.

5. Show that f(x) = (x) is not differentiate at x = 0.

Ans. LHD $= \lim_{x\to 0-} \frac{f(x)-f(0)}{x-0}$

$$\Rightarrow \lim_{h\to 0} \frac{f(0-h)-f(0)}{0-h-0}$$

$$\Rightarrow \lim_{h\to 0} \frac{f(-h)-f(0)}{(-h)}$$

$$\Rightarrow \lim_{h\to 0} \frac{|-h|-|0|}{(-h)} = \lim_{h\to 0} \frac{|-h|}{-h}$$

$\Rightarrow$ -1

RHD $\lim_{x\to 0+} \frac{f(x)-f(0)}{x-0}$

$$= \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0} \frac{f(h)-f(0)}{h}$$

$$= \lim_{h\to 0} \frac{|h|-|0|}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

Since LHD ≠ RHD so that is not differentiable at x = 0.

6. Check the differentiate of the following function at x= $\pi/2$

$$f(x) = \begin{cases} 1+\sin x & when \quad 0 \le x < \pi/2 \\ 2+\left(x-\pi/2\right)^2 & when \quad x \ge \pi/2 \end{cases}$$

Ans. When **x=** $\pi/2$

$$f(x) = f\left(\pi/2\right) = 2+\left(\pi/2-\pi/2\right)^2$$

$$f\left(\pi/2\right) = 2$$

LHD $= \lim_{h\to 0} \frac{f\left(\pi/2-h\right)-f\left(\pi/2\right)}{\left(\pi/2-h\right)-\left(\pi/2\right)}$

$$= \lim_{h\to 0} \frac{f\left(\pi/2-h\right)-f\left(\pi/2\right)}{\frac{\pi}{2}-h-\frac{\pi}{2}}$$

$$= \lim_{h\to 0} \frac{1+\sin\left(\frac{\pi}{2}-h\right)-2}{-h}$$

$$= \lim_{h\to 0} \frac{1+\cos\frac{\pi}{2}-2}{-h}$$

$$= \lim_{h\to 0} \frac{1-\cos(h)}{h} = \lim_{h\to 0} \frac{2\sin^2\left(h/2\right)}{h}$$

$$= \lim_{h\to 0} \left[\frac{\sin\left(h/2\right)}{\left(h/2\right)}\right]^2 .\left(\frac{h}{2}\right) = \frac{1}{2}.(1)^2.0 = 0$$

RHD $= \lim_{h\to 0} \dfrac{f\left(\pi/2+h\right)-f\left(\pi/2\right)}{\frac{\pi}{2}+h-\frac{\pi}{2}}$

$= \lim_{h\to 0} \dfrac{2+\left(\frac{\pi}{2}+h-\frac{\pi}{2}\right)^2-2}{h}$

$= \lim_{h\to 0} \dfrac{h^2}{h} = \lim_{h\to 0} h = 0$

Since LHD = RHD, so the given of f(x) is differentiable at $x=\pi/2$

7. If f(2) = 4 and $f'(2)=1$, then find $\lim_{x\to 2} \dfrac{xf(2)-2f(x)}{x-2}$

Ans. We have $\lim_{x\to 2} \dfrac{xf(x)-2f(x)}{x-2}$

$= \lim_{x\to 2} \dfrac{xf(2)-2f(x)}{x-2}$

$= \lim_{x\to 2} \dfrac{xf(2)-2f(2)+2f(2)-2f(x)}{(x-2)}$

$= \lim_{x\to 2} \dfrac{(x-2)f(2)-2\left[f(x)-f(2)\right]}{(x-2)}$

$= \lim_{x\to 2} \dfrac{(x-2)f(2)}{(x-2)} - 2\lim_{x\to 2} \dfrac{f(x)-f(2)}{(x-2)}$

$= f(2)-2f'(2)$

$\left[f'(2) = \lim_{x\to 2} \dfrac{f(x)-f(2)}{x-2}\right]$

$= 4-2\times 1 = 4-2$

$= 2$

8. Differentiate the function f(x) = e $^{\sin x}$ by first principle.

Ans. $= \dfrac{d}{dx} f(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

$= \dfrac{d}{dx}\left[f(x)\right] = \lim_{h\to o} \dfrac{e^{\sin(x+h)}-e^{\sin x}}{h}$

$$= e^{\sin x} \lim_{h\to 0} \frac{e^{\sin(x+h)-\sin x} - 1}{h}$$

$$= e^{\sin x} \lim_{h\to 0} \left\{\frac{e^{\sin(x+h)-\sin x} - 1}{\sin(x+h)-\sin x}\right\} \times \left\{\frac{\sin(x+h)-\sin x}{h}\right\}$$

$$= e^{\sin x} \lim_{h\to 0} \left\{\frac{e^{\sin(x+h)-\sin x} - 1}{\sin(x+h)-\sin x}\right\} \times \lim_{h\to 0} \left\{\frac{\sin(x+h)-\sin x}{h}\right\}$$

$$= e^{\sin x} \lim_{y\to 0} \left(\frac{e^{y}-1}{y}\right) \times \lim_{h\to 0} \frac{2\sin\left(h/2\right)\cos x + h/2}{2\left(h/2\right)}$$

Where y= sin (x + h) – sin x and when h → 0 ,y→ 0

$$= \frac{d}{dx} f(x) = e^{\sin x} \lim_{y\to 0} \left(\frac{e^{y}-1}{y}\right) \times \lim_{h\to 0} \left(\frac{\sin\left(h/2\right)}{h/2}\right) \times \lim_{h\to 0} \cos\left(x + h/2\right)$$

$$= \frac{d}{dx}(f(x)) = e^{\sin x}(1)\times(1)\times\cos x$$

$$= e^{\sin x} \times \cos x$$

Hence $\frac{d}{dx}\left(e^{\sin x}\right) = e^{\sin x} \times \cos x$.

9. Differentiate the following function w.r.t. x

$$f(x) = \sin\left(x^2+1\right)$$

Ans. Let $y = \sin\left(x^2+1\right)$ putting $u = \left(x^2+1\right)$, we get y= sin u and $u = x^2+1$

$\therefore \frac{dy}{du} = \cos u$ and $\frac{dy}{du} = 2x$

Now $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$= \cos(u) \times 2x$$

$$= 2x\cos\left(x^2+1\right)$$

Hence $\frac{d}{dx}\left\{\sin\left(x^2+1\right)\right\} = 2x\cos\left(x^2+1\right)$

10. Differentiate $\log \sin x^2$ **w. r .t x**

Ans. Let $y = \log\sin x^2$, putting $u = \sin x^2$ and $v = x^2$

$\therefore y = \log u, u = \sin v$ and $v = x^2$

$\therefore \frac{dy}{du} = \frac{1}{u}, \frac{du}{dv} = \cos v$ and $\frac{dv}{dx} = 2x$

Now $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx}$

$$\frac{dy}{dx} = \frac{1}{u} \times \cos v \times 2x$$

$$\frac{dy}{dx} = \frac{1}{\sin v} \times \cos v \times 2x$$

$$\therefore \frac{dy}{dx} = \cot v \times 2x$$

$$\frac{dy}{dx} = 2x \cot x^2$$

Hence $\frac{d}{dx}\left(\log\sin x^2\right) = 2x\cot x^2$

11. Differentiate $y = \tan^{-1}\left(\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}\right)$ **w. r. t. x**

Ans. Put $x = \cos\theta = \theta = \cos^{-1} x$

So $y = \tan^{-1}\left(\frac{\sqrt{1+\cos\theta} - \sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta} + \sqrt{1-\cos\theta}}\right)$

$$y = \tan^{-1}\left(\frac{\sqrt{2}\cos\left(\theta/2\right) - \sqrt{2}\sin\left(\theta/2\right)}{\sqrt{2}\cos\left(\theta/2\right) + \sqrt{2}\sin\theta/2}\right)$$

$$y = \tan^{-1}\left(\frac{\sqrt{2\cos^2\theta/2} - \sqrt{\left(2\sin^2\theta/2\right)}}{\sqrt{2\cos^2\theta/2} + \sqrt{2\sin^2\theta/2}}\right)$$

$$y = \tan^{-1}\left(\frac{1 - \tan\left(\theta/2\right)}{1 + \tan\left(\theta/2\right)}\right)$$

$$y = \tan^{-1}\left[\tan\left(\frac{\pi}{4} - \theta/2\right)\right]$$

$$y = \frac{\pi}{4} - \frac{\theta}{2}$$

$$y = \frac{\pi}{4} - \frac{1}{2}\cos^{-1} x$$

Differentiating w. r. t. x, we get

$$\frac{dy}{dx} = 0 - \frac{1}{2}\left(-\frac{1}{\sqrt{1-x^2}}\right) = \frac{1}{2}\frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}\frac{1}{\sqrt{1-x^2}}$$

12. Differentiate $\sec^{-1}\left(\frac{1}{2x^2-1}\right), 0 < x < \frac{1}{\sqrt{2}}$

Ans. Putting $x = \cos\theta$, we get

$$y = \sec^{-1}\left(\frac{1}{2\cos^2\theta - 1}\right)$$

$$y = \cos^{-1}\left(2\cos^2\theta - 1\right)\left[\therefore \sec^{-1}\left(\frac{1}{x}\right) = \cos^{-1} x\right]$$

$$y = \cos^{-1}(\cos 2\theta)$$

$$y = 2\theta$$

$$y = 2\cos^{-1} x$$

$$\frac{dy}{dx} = \frac{-2}{\sqrt{1-x^2}}$$

13. Differentiate $y = x^3 \sin x$ **w. r. t. x.**

Ans. $y = x^3 \sin x$

$$\frac{dy}{dx} = x^3 \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}\left(x^3\right)$$

$$\frac{dy}{dx} = x^3 \cos x + \sin x \cdot 3x^2$$

$$\therefore \frac{dy}{dx} = x^3 \cos x + 3x^2 \sin x$$

14. Differentiate y = x sin x log x w. r. t. x.

Ans. y = x sin x log x

$$\frac{dy}{dx} = x\sin x\frac{d}{dx}(\log x) + x\log x\frac{d}{dx}(\sin x) + \sin x\log x\frac{d}{dx}(x)$$

$$\left(\frac{dy}{dx}\right) = \not{x}\sin x\left(\frac{1}{\not{x}}\right) + x\log x(\cos x) + \sin x\log x(1)$$

$$\frac{dy}{dx} = \sin x + x\log x\cdot\cos x + \sin x\cdot\log x$$

15. Differentiate $\frac{e^x}{1+\sin x}$

Ans. Let y = $\frac{e^x}{1+\sin x}$

$$\frac{dy}{dx} = \frac{(1+\sin x)\frac{d}{dx}(e^x) - e^x\frac{d}{dx}(1+\sin x)}{(1+\sin x)^2}$$

$$\frac{dy}{dx} = \frac{(1+\sin x)e^x - e^x(0+\cos x)}{(1+\sin x)^2}$$

$$\frac{dy}{dx} = \frac{e^x(1+\sin x - \cos x)}{(1+\sin x)^2}$$

16. Differentiate $y = Log\left(x+\sqrt{a^2+x^2}\right)$ **w. r. t. x.**

Ans. Let $y = Log\left(x+\sqrt{a^2+x^2}\right)$

$$\therefore \frac{dy}{dx} = \frac{1}{x+\sqrt{a^2+x^2}}\cdot\frac{d}{dx}\left(x+\sqrt{a^2+x^2}\right)$$

$$\frac{dy}{dx} = \frac{1}{x+\sqrt{a^2+x^2}}\times\left\{1+\frac{1}{2\sqrt{a^2+x^2}}\cdot\frac{d}{dx}\left(a^2+x^2\right)\right\}$$

$$\frac{dy}{dx} = \frac{1}{x+\sqrt{a^2+x^2}}\left\{1+\frac{1}{\not{2}\sqrt{a^2+x^2}}\times\not{2}x\right\}$$

$$\therefore \frac{dy}{dx} = \frac{1}{x+\sqrt{a^2+x^2}}\left\{1+\frac{x}{\sqrt{a^2+x^2}}\right\}$$

$$\frac{dy}{dx} = \frac{1}{x+\sqrt{a^2+x^2}}\left\{\frac{\sqrt{a^2+x^2}+x}{\sqrt{a^2+x^2}}\right\}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{a^2+x^2}}$$

17. Differentiate $(\sin x)^x$ w. r. t. x.

Ans. Let y = $(\sin x)^x$. Taking log both sides.

log y = x log (sin x).

Differentiating both sides w. r. t. x

$$\frac{1}{y}\frac{dy}{dx} = x\frac{d}{dx}(\log(\sin x)) + \log\sin x\frac{d}{dx}(x)$$

$$\frac{1}{y}\frac{dy}{dx} = x\cdot\frac{1}{\sin x}\cdot\cos x + \log\sin x(1)$$

$$\frac{1}{y}\frac{dy}{dx} = x\cot x + \log\sin x$$

$$\frac{dy}{dx} = y\left[x+\cot x+\log_e\sin x\right]$$

$$\frac{dy}{dx} = (\sin x)^x\left[x\cot x+\log\sin x\right]$$

18. Find $\left({}^{dy}\!/_{dx}\right)$, when $x=\sin^{-1}\left(\frac{2t}{1+t^2}\right), y=\cos^{-1}\left(\frac{1-t^2}{1+t^2}\right)$

Ans. Let $t=\tan\theta$

$$\therefore x=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)=\sin^{-1}(\sin 2\theta)$$

$$x=2\theta$$

$$\frac{dx}{d\theta}=2$$

$$y=\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$y=\cos^{-1}(\cos 2\theta)$$

$$y = 2\theta$$
$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2}{2} = 1$$
$$\therefore \frac{dy}{dx} = 1$$

19. If $x = at^2$ and y = 2at then find dy/dx

Ans. Since $x = at^2$ and y = 2at

$$\therefore \frac{dx}{dt} = 2at, \frac{dy}{dt} = 2a$$
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$
$$\frac{dy}{dx} = \frac{2a}{2at}$$
$$\frac{dy}{dx} = \frac{1}{t}$$

20. Differentiate $\cos^{-1}\sqrt{x}$ w. r. t. $\sqrt{1-x}$

Ans. y = $\cos^{-1}\sqrt{x}$

$$\therefore \frac{dy}{dx} = \frac{-1}{\sqrt{1-x}} \times \frac{1}{2\sqrt{x}}$$
$$z = \sqrt{1-x}$$
$$\frac{dz}{dx} = \frac{-1}{2\sqrt{1-x}}$$
$$\frac{dy}{dz} = \frac{(dy/dx)}{(dz/dx)} = \left(\frac{-1}{\sqrt{1-x}} \times \frac{1}{2\sqrt{x}}\right) \times \left(-2\sqrt{1-x}\right)$$
$$\left(\frac{dy}{dz}\right) = \frac{1}{\sqrt{x}}$$

21. Differentiate x^x w. r. t. $\log_e x$

Ans. y= x^x and z= $\log_e x$

Taking log

Log y = x log x

Now differentiating both sides

$$\frac{1}{y}\frac{dy}{dx} = x \times \frac{1}{x} + \log x(1)$$

$$\frac{1}{y}\frac{dy}{dx} = 1 + \log x$$

$$\frac{dy}{dx} = y(1+\log x)$$

$$\left(\frac{dy}{dx}\right) = x^x(1+\log x)$$

$$\frac{dz}{dx} = \frac{1}{x}$$

$$\frac{dy}{dz} = \frac{(dy/dx)}{(dz/dx)}$$

$$\frac{dy}{dz} = \frac{x^x(1+\log x)}{1/x}$$

$$\frac{dy}{dz} = x \cdot x^x(1+\log_e x)$$

$$\frac{dy}{dx} = x^{x+1}[\log_e(ex)]$$

22. Find (dy/dx) if

i) $x^y = y^x$

ii) $y = \sqrt{x+\sqrt{x+\sqrt{x+--\infty}}}$

i) Given $x^y = y^x$ Taking log both sides, we get $y\log_e x = x\log_e y$

Now differentiating both sides w. r. t. x.

$$y\left(\frac{1}{x}\right) + \log_e x\left(\frac{dy}{dx}\right) = x \times \frac{1}{y}\frac{dy}{dx} + \log_e y(1)$$

$$\frac{y}{x} + \log_e x\left(\frac{dy}{dx}\right) = \frac{x}{y}\left(\frac{dy}{dx}\right) + \log_e y$$

$$\frac{y}{x} - \log_e y = \frac{dy}{dx}\left(\frac{x}{y} - \log_e x\right)$$

$$\frac{dy}{dx} = \frac{\frac{y}{x} - \log_e y}{\frac{x}{y} - \log_e x}$$

$$\frac{dy}{dx} = \frac{y(y - x\log_e y)}{x(x - y\log_e x)}$$

ii) Let $y=\sqrt{x+\sqrt{x+\sqrt{x+--\infty}}}$

$\therefore y=\sqrt{x+y}$

Squaring both sides, we get $y^2=x+y$

Now differentiating both sides w. r. t. x

$2y\left(\frac{dy}{dx}\right)=1+\left(\frac{dy}{dx}\right)$

$\frac{dy}{dx}(2y-1)=1$

$\therefore \frac{dy}{dx}=\left(\frac{1}{2y-1}\right)$

23. Find $\left(\frac{dy}{dx}\right) if ax^2+2hxy+by^2+2gx+2fy+c=0$

Ans. Given $ax^2+2hxy+by^2+2gx+2fy+c=0$

Differentiating w. r. t. we get:-

$=2ax+2hy(1)+2hx\left(\frac{dy}{dx}\right)+2by\frac{dy}{dx}+2g+2f\frac{dy}{dx}+0=0$

$=2ax+2hy+2g+\frac{dy}{dx}(2hx+2by+2f)=0$

$=\not{2}(hx+by+f)\frac{dy}{dx}=\not{2}(ax+hy+g)$

$\Rightarrow \frac{dy}{dx}=\frac{-(ax+by+g)}{(hx+by+f)}$

24.If y = A sin x + B cos x, prove that $\frac{d^2y}{d^2x}+y=0$

Ans. Given y = A sin x + B cos x. Differentiating both sides w. r. t. x

$\left(\frac{dy}{dx}\right)=A\cos x-B\sin x$

Again differentiating w. r. t. x

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(A\cos x - B\sin x)$$

$$\frac{d^2y}{dx^2} = -A\sin x - B\cos x$$

$$\frac{d^2y}{dx^2} = -(A\sin x + B\cos x)$$

$$\frac{d^2y}{dx^2} = -y$$

$$\Rightarrow \frac{d^2y}{dx^2} + y = 0$$

Hence proved.

25. If $y = \sin^{-1} x$, then show that $\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$

Ans. Given $y = \sin^{-1} x$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \sqrt{1-x^2}\,\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{d}{dx}\left\{\sqrt{1-x^2}\,\frac{dy}{dx}\right\} = 0$$

$$\Rightarrow \sqrt{1-x^2}\,\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot\frac{d}{dx}\sqrt{1-x^2} = 0$$

$$\Rightarrow \sqrt{1-x^2}\,\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(\frac{1}{\not{2}\left(\sqrt{1-x^2}\right)} \times -\not{2}x\right) = 0$$

$$\Rightarrow \sqrt{1-x^2}\,\frac{d^2y}{dx^2} - \frac{dy}{dx}\left(\frac{x}{\sqrt{1-x^2}}\right) = 0$$

$$\Rightarrow \left(1-x^2\right)\left(\frac{d^2y}{dx^2}\right) - x\left(\frac{dy}{dx}\right) = 0$$

Hence proved.

26. Discuss the applicability of Rolle's theorem for the following function on the indicated interval $f(x) = e^x \sin x$**,**

$\forall x \in [0, \pi]$

Ans. i) Given function is a product of two continuous function e^x and sin x. Hence f(x) is continuous for every value of x. Hence f(x) is continuous in $[0, \pi]$ interval.

ii) $f'(x) = e^x \sin x + e^x \cos x$

. $f'(x) = e^x(\sin x + \cos x)$

Which is defined for every value of x in the interval $(0, \pi)$. Hence f (x) is differentiable in this interval.

iii) f (0) = 0 = f (π).

Hence the function satisfies all the three conditions of Rolle's theorem. Now $f'(x) = 0$

$\Rightarrow \cos x + \sin x = 0$

$\Rightarrow \cos x = -\sin x$

$\Rightarrow -\tan x = 1$

$\Rightarrow x = \pi - \frac{\pi}{4}$

$\Rightarrow x = \left(\frac{3\pi}{4}\right)$

Clearly $\left(\frac{3\pi}{4}\right) \varepsilon (0, \pi), \{\therefore e^x \neq 0\}$

Hence the interval $(0, \pi)$ contains one point $c = \frac{3\pi}{4}$ for which f '(c) =0.

Hence the Rolle's Theorem is verified.

27. Verify Rolle's Theorem for the function $f(x) = x(x-3)^2, 0 \le x \le 3$.

Ans. We have,

$$f(x) = x(x-3)^2$$

$$f(x) = x(x^2 - 6x + 9)$$

$$f(x) = x^3 - 6x^2 + 9x$$

We know that a polynomial function is every where differentiable and so continuous

also. So f (x) is continuous in [0, 3] and differentiable in (0, 3).

Also f (0) = 0 = f (3). Thus, all the conditions of Rolle's theorem are satisfied. Now we

have to show that there exist $c \in (0,3)$, Such that f ' (c) =0

We have

$f(x) = x^3 - 6x^2 + 9x$

$f'(x) = 3x^2 - 12x + 9$

$\therefore f'(x) = 0$

$\Rightarrow 3x^2 - 12x + 9 = 0$

$\Rightarrow 3x^2 - 3x - 9x + 9 = 0$

$\Rightarrow 3x(x-1) - 9(x-1) = 0$

$\Rightarrow (x-1)(3x-9) = 0$

$\Rightarrow x = 1,3$

Thus c = 1 $\in (0,3)$ such that f ' (c) = 0. Hence rolles theorem is verified.

28. Verify Lagrange's mean values theorem for the following function on the indicated intervals

$f(x) = \log_e x \forall x \in [1,2]$

Ans. Since $f(x) = \log_e x$ is differentiable and so continuous for all x>0. So f (x) is continuous

on [1,2] and differentiable on (1,2). Thus both the conditions of langrange's mean value

Theorem is satisfied. Hence there exist some $c \in (1,2)$ such that :

$$f'(c) = \frac{f(2) - f(1)}{2 - 1}$$

$$f(x) = \log_e x$$

$$\Rightarrow f'(x) = \frac{1}{x}$$

$$f(2) = \log_e 2, f(1) = \log_e 1 = 0$$

$$\therefore f'(x) = \frac{f(2) - f(1)}{2 - 1}$$

$$\frac{1}{x} = \frac{\log_e 2 - 0}{1}$$

$$\Rightarrow \frac{1}{x} = \log_e 2$$

$$\Rightarrow x = \frac{1}{\log_e 2} = \log_2 e$$

$$Now 2 < e < 4$$

$$\Rightarrow \log_2 2 < \log_2 e < \log_2 4$$

$$\Rightarrow 1 < \log_2^e < 2$$

$$c = \log_2 e \in (1, 2)$$

$$f'(c) = \frac{f(2) - f(1)}{2 - 1}$$

Hence Lagrange's mean value theorem is verified.

29. Check the validity of Lagrange mean value theorem for the following function

$f(x) = |x|, \forall x \in [-1, 1]$

Ans. We know that $f(x) = |x|$ is continuous every where. Hence the function $f(x) = |x|$ is also continuous in [-1, 1] interval. But $f(x) = |x|$ is not differentiable at x = 0. Hence the function $f(x) = |x|$ is not differentiable in interval (-

Chapter – 2

Application of Derivatives

1. The distance travelled by a particle in t second moving along a straight line is given by $s = t^3 + 6t^2 + 5t + 6$. Find velocity and acceleration at t= 5 second.

Ans. Given $s = t^3 + 6t^2 + 5t + 6$

Velocity $v = \frac{ds}{dt}$

$v = 3t^2 + 12t + 5$

Velocity at t =5 second

$v = 3(5)^2 + 12(5) + 5$

$v = 140 m/\sec.$

Now acceleration $a = \frac{dv}{dt}$

$a = \frac{dv}{dt}$

$= \frac{d}{dt}\left(3t^2 + 12t + 5\right)$

$a = 6t + 12$

Acceleration at t = 5 sec.

a= 6(5)+12

=30+12

= 42 m/sec^2.

2. If a particle is moving along a straight line according to the formula $s = t^3 - 6t^2 - 15t$. Then find the time interval for which velocity is –ve and acceleration is positive.

Ans. Given $s = t^3 - 6t^2 - 15t$

Velocity $v = \frac{ds}{dt} = 3t^2 - 12t - 15$

And acceleration $a = \frac{dv}{dt}$

$a = 6t - 12$

Now $v < 0$ and $a > 0$

$\Rightarrow 3t^2 - 12t - 15 < 0 \quad and \quad 6t - 12 > 0$

$\Rightarrow 3\left(t^2 - 4t - 5\right) < 0 \quad and \quad 6(t-2) > 0$

$\Rightarrow \left(t^2 - 4t - 5\right) < 0 \quad and \quad (t-2) > 0$

$\Rightarrow (t-5)(t+1) < 0 \quad and \quad (t-2) > 0$

$\Rightarrow -1 < t < 5 \quad and \quad t > 2$

$\Rightarrow 2 < t < 5$

Hence after 2 second and before 5 second velocity will be negative and acceleration will be positive.

3. Find the points on curve $x^2 + y^2 - 2x - 3 = 0$ where tangent is

i) Parallel to x – axis

ii) Perpendicular to x – axis

iii) Makes equal angle with both axes

Ans. Given curve is $x^2 + y^2 - 2x - 3 = 0$ __________________(1)

Differentiating with respect to x we get

$\Rightarrow 2x + 2y\frac{dy}{dx} - 2 = 0$

$\Rightarrow x + y\frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = \left(\frac{1-x}{y}\right)$ ________________(2)

i) Tangent is parallel to x – axis

$\Rightarrow \frac{dy}{dx} = \tan 0 = 0$

$\therefore \frac{1-x}{y} = 0$

$\Rightarrow x = 1$

Substituting x = 1 in equation (1)

$1 + y^2 - 2 - 3 = 0$

$y^2 = 4$

$y = \pm 2$

Hence required points are (1,2) and (1, -2).

ii) Tangent is perpendicular to x - axis

$\Rightarrow \frac{dy}{dx} = \tan 90°$
$= \infty$
So from equation (2)
$\Rightarrow \frac{1-x}{y} = \infty = \frac{1}{0}$
$\Rightarrow y = 0$
Substituting y = 0 in equation (1)
$= x^2 + 0 - 2x - 3 = 0$
$= x^2 - 2x - 3 = 0$
$= (x-3)(x+1) = 0$
$= x = -1, 3$
Hence required points are (-1, 0) and (3, 0).
iii) Making equal angle with both axes
$\Rightarrow \frac{dy}{dx} = \tan 45° = 1$
From equation (2)
$= \frac{1-x}{y} = 1$
$\Rightarrow y = 1 - x$
Substituting y = (1 – x) in equation (1)
$= x^2 + (1-x)^2 - 2x + 3 = 0$
$= x^2 + 1 + x^2 - 2x - 2x - 3 = 0$
$= 2x^2 - 4x - 2 = 0$
$= x^2 - 2x - 1 = 0$
$\Rightarrow x = 1 \pm \sqrt{2}, y = 1 - (1 \pm \sqrt{2}) = \mp\sqrt{2}$
Hence required points are $(1+\sqrt{2}, -\sqrt{2})$ *and* $(1-\sqrt{2}, \sqrt{2})$

4. Find tangent and normal of the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, on $(a\cos\theta, b\sin\theta)$.

Ans. Given curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Differentiating w. r. t. x, we get

$$= \frac{2x}{a^2} + \frac{2y}{b^2}\left(\frac{dy}{dx}\right) = 0$$

$$\Rightarrow \left(\frac{dy}{dx}\right) = -\frac{xb^2}{ya^2}$$

$$\left(\frac{dy}{dx}\right)_{(a\cos\theta, b\sin\theta)} = \frac{-a\cos\theta \times b^2}{b\sin\theta \times a^2} = \frac{-b\cos\theta}{a\sin\theta}$$

$$= -\frac{1}{\left(\frac{dy}{dx}\right)_{(a\cos\theta, b\sin\theta)}} = \frac{a\sin\theta}{b\cos\theta}$$

Hence equation of tangent at $(a\cos\theta, b\sin\theta)$ is

$$= y - b\sin\theta = \frac{-b\cos\theta}{a\sin\theta}(x - a\cos\theta)$$

$$= ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta$$

$$= bx\cos\theta + ay\sin\theta = ab\left(\cos^2\theta + \sin^2\theta\right)$$

$$\Rightarrow bx\cos\theta + ay\sin\theta = ab$$

Equation of normal at $(a\cos\theta, b\sin\theta)$

$$= y - b\sin\theta = \frac{a\sin\theta}{b\cos\theta}(x - a\cos\theta)$$

$$= by\cos\theta - b^2\sin\theta\cos\theta = ax\sin\theta + a^2\sin\theta\cos\theta$$

$$= ax\sin\theta - by\cos\theta = \left(a^2 - b^2\right)\sin\theta\cos\theta$$

$$\Rightarrow ax\sec\theta - by\operatorname{cosec}\theta = \left(a^2 - b^2\right)$$

5. A balloon, which always remain spherical has a variable diameter $\frac{3}{2}(2x+3)$. Determine the rate of change of volume with respect to x.

Ans. Diameter of balloon = $\frac{3}{2}(2x+3)$

$$\therefore Radius = \frac{3}{4}(2x+3)$$

Volume of a sphere (V) = $= \frac{4}{3}\pi r^3$

$$= \frac{4}{3}\pi\left[\frac{3}{4}(2x+3)\right]^3$$

$$V = \frac{9\pi}{16}(2x+3)^3$$

Rate of change of volume $\frac{dv}{dx} = \frac{9\pi}{16} \cdot 3 \cdot (2x+3)^2 \times 2$

$$\frac{dv}{dx} = \frac{27\pi}{8}(2x+3)^2$$

6. A stone is dropped in to a quite lake and waves move in a circle at a speed of 3.5 cm/sec. At the instant when the radius of the circular wave is 7.5 cm, how fast is the enclosed area increasing?

Ans. Let r be the radius and A be the area of the circular wave at any time t then

$$A = \pi r^2 \, and \, \frac{dr}{dt} = 3.5 cm/\sec.$$

$$\frac{dA}{dt} = \pi\left(2r\frac{dr}{dt}\right)$$

$$\Rightarrow \frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

$$\Rightarrow \frac{dA}{dt} = 2\pi r(3.5)$$

$$\Rightarrow \frac{dA}{dt} = 7\pi r$$

$$\Rightarrow \frac{dA}{dt} = 7\pi(7.5)$$

$$= 52.5\pi cm^2/\sec.$$

7. Sand is pouring from a pipe at the rate of $12 cm^3/\sec.$ The falling sand forms a cone on the ground in such a way that the height of the cone is always one – sixth of the radius of the base. How fast is the height of the sand – cone increasing when the height is 4 cm?

Ans. Let r be the radius, h be the height and v be the volume of the sand – cone at any time t.

Then.

$$V=\frac{1}{3}\pi r^2 h$$

$$\Rightarrow V=\frac{1}{3}\left(36h^2\right)h$$

$$\Rightarrow V=12\pi h^3$$

$$\Rightarrow \frac{dV}{dt}=36\pi h^2\left(\frac{dh}{dt}\right)$$

$$\Rightarrow 12=36\pi h^2\left(\frac{dh}{dt}\right)$$

$$\Rightarrow \frac{dh}{dt}=\frac{1}{3\pi h^2}$$

$$\Rightarrow \left(\frac{dh}{dt}\right)_{h=4}=\frac{1}{3\pi(4)^2}=\frac{1}{48\pi}$$

Thus, the height of the sand – cone is increasing at the rate of $\frac{1}{48\pi} cm/\sec.$

8. Find all the points of local maxima and minima of the function $f(x)=x^3-6x^2+9x-8$

Ans. Given y = $f(x)=x^3-6x^2+9x-8$. Then $\frac{dy}{dx}=3x^2-12x+9$

For maxima and minima $\frac{dy}{dx}=0$

$$\Rightarrow 3x^2-12x+9=0$$

$$\Rightarrow x^2-4x+3=0$$

$$\Rightarrow x^2-3x-x+3=0$$

$$\Rightarrow x(x-3)-1(x-3)=0$$

$$\Rightarrow (x-1)(x-3)=0$$

$$\Rightarrow x=1,3$$

Now we have to check that whether these points are the points of maxima or minima. So

$$\frac{d^2y}{dx^2} = 6x - 12$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=1} = 6(1) - 12 = 6 < 0$$

Hence at x = 1 the given function has maximum value and the value is

$$f(x) = (1)^3 - 6(1)^2 + 9(1) - 8$$

$$f(x) = 1 - 6 + 9 - 8$$

$$f(x) = -4$$

Now $\left(\frac{d^2y}{dx^2}\right)_{x=3} = 6(3) - 12)$

$$\left(\frac{d^2y}{dx^2}\right)_{x=3} = 18 - 12$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=3} = 6 > 0$$

Hence the function has minimum value at x = 3. The minimum value of the function at x = 3 is

$$f(x) = (3)^3 - 6(3)^2 + 9(3) - 8$$

$$f(x) = \cancel{27} - \cancel{54} + \cancel{27} - 8$$

$$f(x) = -8$$

9. Find the maxima and minima of the function $f(x) = (x-1)^2 e^x$.

Ans. Given $y = (x-1)^2 e^x$

$$\frac{dy}{dx} = 2(x-1)e^x + (x-1)^2 e^x$$

For maxima and minima $\frac{dy}{dx} = 0$

$$\Rightarrow \left[2(x-1)+(x-1)^2\right]e^x = 0$$

$$\Rightarrow \left[\cancel{2x}-2+x^2-\cancel{2x}+1\right]e^x = 0$$

$$\Rightarrow \left[x^2-1\right]e^x = 0$$

$$\therefore e^x \neq 0, \left(x^2-1=0\right)$$

$$\Rightarrow x = \pm 1$$

Now $\dfrac{d^2y}{dx^2} = 2e^x + 2(x-1)e^x + 2(x-1)e^x + (x-1)^2 e^x$

$$\frac{d^2y}{dx^2} = e^x\left[2+4(x-1)+(x-1)^2\right]$$

Now for x= 1

$$\left(\frac{d^2y}{dx^2}\right)_{x=1} = e'\left[2+4\times 0+0\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=1} = 2e < 0$$

Hence the value of the function is minimum at x = 1 and the minimum value is

$$f(x) = (1-1)^2 e^1 = 0$$

Now for x = -1

$$\left(\frac{d^2y}{dx^2}\right)_{x=-1} = e^{-1}\left[2+4(-2)+4\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=-1} = e^{-1}\left[2-8+4\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=-1} = \frac{-2}{e} < 0$$

Hence the value of the function is maximum at x = -1. The maximum value is

$$= (-1-1)^2 e^{-1} = 4e^{-1} = \frac{4}{e}$$

10. Prove that the maximum value of $\left(\frac{1}{x}\right)e^{\frac{1}{e}}$

Ans. Let $y=\left(\frac{1}{x}\right)^{x}$ Taking log both sides.

Log y = $x\log\left(\frac{1}{x}\right)$

$\log y=-x\log x$

Log z = z(let)

Here the maximum or minimum value of will also be maximum or minimum value of z

so

$$\frac{dz}{dx}=-x\cdot\frac{1}{x}-\log x$$

$$\frac{dz}{dx}=-(1+\log x)$$

And maxima or minima

$$\frac{dz}{dx}=0$$

$$\Rightarrow-(1+\log x)=0$$

$$\Rightarrow x=e^{-1}$$

$$\Rightarrow x=\frac{1}{e}$$

Now

$$\frac{d^2z}{d^2x}=-\frac{1}{x}$$

$$\left(\frac{d^2y}{d^2x}\right)_{x=\frac{1}{e}}=-\frac{1}{1/e}=-e<0$$

Hence the function has maximum value at $x=\frac{1}{e}$

And maximum value is $e^{\frac{1}{e}}$

Hence proved.

Chapter – 3

Integrals

1. Integrate the following – $\int \frac{dx}{\sin^2 x \cos^2 x}$

Ans. $\int \frac{dx}{\sin^2 x \cos^2 x} = \int \frac{(\sin^2 x + \cos^2 x)dx}{\sin^2 x \cos^2 x}$

$$= \int \frac{\sin^2 x .dx}{\sin^2 x \cos^2 x} + \int \frac{\cos^2 x dx}{\sin^2 x \cos^2 x}$$

$$= \int \frac{1}{\cos^2 x} dx + \int \frac{1}{\sin^2 x} dx$$

$$= \int \sec^2 x dx + \int \cos ec^2 x dx$$

$$= \tan x - \cot x + c$$

2. Integrate $\int \frac{dx}{(e^x + e^{-x})^2}$

Ans. $\int \frac{dx}{(e^x + e^{-x})^2} = \int \frac{dx}{4\left(\frac{e^x + e^{-x}}{2}\right)^2}$

$$= \frac{1}{4} \int \frac{dx}{\cosh^2 x}$$

$$= \frac{1}{4} \int \sec h^2 x dx$$

$$\left[\frac{e^x + e^{-x}}{2} = \cosh x\right]$$

$$= \frac{1}{4} \tanh x + c$$

3. Integrate the following –

(I) $\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx$ **II)** $\int \frac{\cos(\log x)}{x} dx$

Ans. i) $\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx$ ________________ (1)

Let $\sqrt{x} = t$

$= \frac{1}{2\sqrt{x}}dx = dt$

$\Rightarrow \frac{1}{\sqrt{x}}dx = 2dt$

From eg. (1)

$I = 2\int \sin t dt$

$= 2(-\cos t) + c$

$= -2\cos t + c$

$= -2\cos\left(\sqrt{x}\right) + c$

i) Let $I = \int \frac{\cos(\log x)}{x}dx - - - - - (1)$

Let log x = t

$\frac{1}{x}dx = dt$

From (1)

$I = \int \cos t.dt$

$= \sin t + c$

$= \sin(\log x) + c$

4. Integrate the following – $\int \frac{\sin x \cos x dx}{a\cos^2 x + b\sin^2 x}$

Ans. Let I = $\int \frac{\sin x \cos x dx}{a\cos^2 x + b\sin^2 x}$

. Let $= a\cos^2 x + b\sin^2 x = t$

$= (-2a\cos x \sin x + 2b\sin x \cos x).dx = dt$

$= 2(b-a)\sin x \cos x dx = dt$

$\Rightarrow \sin x \cos x dx = \frac{dt}{2(b-a)}$

$\therefore I = \int \frac{dt}{2(b-a).t}$

$= \frac{1}{2(b-a)}\log t + c$

$I = \frac{1}{2(b-a)}\log\left(a\cos^2 x + b\sin^2 x\right) + c$

5. $\int \frac{1}{4+5\sin x}dx$

Ans. Let I = $\int \frac{1}{4+5\sin x}dx$

Since $\sin x = \frac{2\tan\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}$

$$\therefore I = \int \frac{dx}{4+5\left(\frac{2\tan\left(x/2\right)}{1+\tan^2\left(x/2\right)}\right)}$$

$$= \int \frac{dx\left(1+\tan^2\left(x/2\right)\right)}{\left(4+4\tan^2\left(x/2\right)+10\tan\left(x/2\right)\right)}$$

$$= \frac{1}{4}\int \frac{\sec^2\left(x/2\right)dx}{1+\tan^2\left(x/2\right)+\frac{5}{2}\tan\left(x/2\right)}$$

Let $\tan\left(\frac{x}{2}\right) = t$

$$= \frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx = dt$$

$$\therefore I = \frac{1}{4}\int \frac{2dt}{1+t^2+\frac{5}{2}t}$$

$$= \frac{1}{2}\int \frac{dt}{t^2 + \frac{5}{2}t + 1}$$

$$= \frac{1}{2}\int \frac{dt}{t^2 + \frac{5}{4}t + \frac{5}{4}t + \frac{25}{16} - \frac{25}{16} + 1}$$

$$= \frac{1}{2}\int \frac{dt}{(t + 5/2)^2 - (9/16)}$$

$$= \frac{1}{2}\frac{1}{2\left(\frac{3}{4}\right)}\log\left\{\frac{\left(t + \frac{5}{4}\right) - \left(\frac{3}{4}\right)}{\left(t + \frac{5}{4}\right) + \left(\frac{3}{4}\right)}\right\} + C$$

$$= \frac{1}{3}\log\left\{\frac{t + 1/2}{t + 2}\right\} + C$$

$$= \frac{1}{3}\log\left\{\frac{2t + 1}{2t + 4}\right\} + C$$

$$= \frac{1}{3}\log\left\{\frac{2\tan\left(x/2\right) + 1}{2\tan\left(x/2\right) + 4}\right\} + C$$

6. Integrate $\int \frac{\sin x}{\sqrt{1 + \sin x}}dx$

Ans. Let I = $\int \frac{\sin x}{\sqrt{1 + \sin x}}dx$

$$= \int \frac{(1 + \sin x) - 1}{\sqrt{1 + \sin x}}dx$$

$$= \int \frac{(1 + \sin x)}{\sqrt{1 + \sin x}} - \int \frac{1}{\sqrt{1 + \sin x}}dx$$

$$= \int \sqrt{1 + \sin x} - \int \frac{1}{\sqrt{1 + \sin x}}$$

$$= \left[(1 + \sin x) = \left\{\cos\frac{x}{2} + \sin\left(\frac{x}{2}\right)\right\}^2\right]$$

$$\therefore I = \int \left\{\cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right)\right\}dx - \int \frac{1}{\cos\frac{x}{2} + \sin\left(\frac{x}{2}\right)}dx$$

$$I = 2\sin\left(\frac{x}{2}\right) - 2\cos\left(\frac{x}{2}\right) - \int \frac{1}{\sqrt{2}} \operatorname{cosec}\left(\frac{\pi}{4} + \frac{x}{2}\right) dx$$

$$I = 2\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right) - \frac{1}{\sqrt{2}} \times 2\log\tan\left(\frac{\pi}{8} + \frac{x}{4}\right) + c$$

$$I = 2\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right) - \sqrt{2}\log\tan\left(\frac{\pi}{8} + \frac{x}{4}\right) + c$$

7. Integrate $\int x^2 \sin x dx$

Ans. Here x^2 is taken as first function and sin x as second function (according to ILATE)

Now let I = $\int x^2 \sin x dx$

Integrating by parts

$$\therefore I = x^2 \{\int \sin x dx\} - \int \left\{\frac{d}{dx}(x)^2 \int \sin x dx\right\} dx$$

$$I = -x^2 \cos x - \int 2x(-\cos x) dx$$

$$I = -x^2 \cos x + 2\int x \cos x dx$$

$$I = -x^2 \cos x + 2\left[x\int \cos x dx - \int \left\{\frac{d}{dx}(x).\int \cos x dx\right\} dx\right]$$

$$I = -x^2 \cos x + 2\left[x \sin x - \int \sin x dx\right]$$

$$I = -x^2 \cos x + 2x \sin x + 2\cos x + c$$

8. Integrate $\int \log\left(x + \sqrt{x^2+1}\right) dx$

Ans. Let I = $\int \log\left(x + \sqrt{x^2+1}\right) dx$

$$I = \log\left(x + \sqrt{x^2+1}\right)\int 1.dx - \int \left\{\frac{d}{dx}\log\left(x + \sqrt{x^2+1}\right)\int dx\right\} dx$$

$$I = \log\left(x + \sqrt{x^2+1}\right) \times x - \int \frac{1}{x + \sqrt{x^2+1}} \times \left(1 + \frac{\cancel{2}x}{\cancel{2}\left(\sqrt{x^2+1}\right)}\right) x dx$$

$$I = x\log\left(x + \sqrt{x^2+1}\right) - \int \frac{1}{x + \cancel{\sqrt{x^2+1}}} \left\{\frac{\cancel{\sqrt{x^2+1}+x}}{\sqrt{x^2+1}}\right\} x dx$$

$$I = x\log\left(x + \sqrt{x^2+1}\right) - \int \frac{x}{\sqrt{x^2+1}} dx -----(1)$$

Let

$$I_1 = \int \frac{x}{\sqrt{x^2+1}} dx$$

$$x^2 + 1 = t^2$$

$$2xdx = 2tdt$$

$$xdx = \frac{tdt}{t}$$

$$\therefore I_1 = \int \frac{\not{t} dt}{\not{t}} = t + c$$

$$\therefore I_1 = \sqrt{x^2+1} + c --- (2)$$

From (1) & (2)

$$I = x \log\left(x + \sqrt{x^2+1} \right) - \sqrt{x^2+1} + c$$

9. **Solve** $\int \frac{3x-2}{(x-1)^2 (x+1)(x+2)}$

Ans. Let $\int \frac{3x-2}{(x-1)^2 (x+1)(x+2)} = \frac{A_1}{(x-1)} + \frac{A_2}{(x-1)^2} + \frac{A_3}{(x+1)} + \frac{A_4}{(x+2)}$

$$\Rightarrow 3x - 2 = A_1 (x-1)(x+1)(x+2) + A_2 (x+1)(x+2) + A_3 (x-1)^2 (x+2) + A_4 (x-1)^2 (x+1)$$

Putting x= 1

$$3(1) - 2 = A_2 (1+1)(1+2)$$

$$1 = 6A_2$$

$$\Rightarrow A_2 = \tfrac{1}{6}$$

Putting x = -1

$$3(-1) - 2 = A_3 (-1-1)^2 (-1+2)$$

$$-5 = A_3 (4)(1)$$

$$\Rightarrow A_3 = \frac{-5}{4}$$

Putting x = -2

$$3(-2)-2=A_4(-2-1)^2(-2+1)$$
$$-8=A_4(9)(-1)$$
$$\Rightarrow A_4=\frac{8}{9}$$

Putting x= 0

$$-2=A_1(-1)(1)(2)+A_2(1)(2)+A_3(1)(2)+A_4(1)(1)$$
$$-2=-2A_1+2A_2+2A_3+A_4$$
$$-2=-2A_1+2\left(\frac{1}{6}\right)+2\left(\frac{-5}{4}\right)+\frac{8}{9}$$
$$-2=2A_1+\frac{1}{3}-\frac{5}{2}+\frac{8}{9}$$
$$2A_1=\frac{1}{3}-\frac{5}{2}+\frac{8}{9}+\frac{2}{1}$$
$$2A_1=\frac{6-45+16+36}{18}$$
$$2A_1=\frac{13}{18}$$
$$\Rightarrow A_1=\frac{13}{36}$$
$$\therefore \int\frac{3x-2}{(x-1)^2(x+1)(x+2)}=\int\frac{13}{36(x-1)}+\int\frac{1}{6(x-1)^2}-\int\frac{5}{4(x+1)}+\int\frac{8}{9(x+2)}$$
$$=\frac{13}{36}\log(x-1)+\frac{1}{6}\left(\frac{-1}{(x-1)}\right)-\frac{5}{4}\log(x+1)+\frac{8}{9}\log(x+2)$$
$$=\frac{13}{36}\log(x-1)-\frac{1}{6(x-1)}-\frac{5}{4}\log(x+1)+\frac{8}{9}\log(x+2)$$

10. Evaluate $\int\frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)}dx$

Ans. Let $\int\frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)}=1+\frac{A}{(x-4)}+\frac{B}{(x-5)}+\frac{C}{(x-6)}$ ---------(1)

Then

$$(x-1)(x-2)(x-3)=(x-4)(x-5)(x-6)+A(x-5)(x-6)+B(x-4)(x-6)+C(x-4)(x-5)$$

Putting x = 4

(3) (2) (1) = A (-1) (-2)

6 = 2A

A = 3

Putting x = 5
(4) (3) (2) = B (1) (-1)
-24 = B
Putting x = 6
(5) (4) (3) = C (2) (1)
60 = 2C
= C = 30
From (1)

$$\frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)} = 1 + \frac{3}{x-4} - \frac{24}{x-5} + \frac{30}{x-6}$$

$$\int \frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)} dx = \int 1.dx + 3\int \frac{dx}{x-4} - 24\int \frac{dx}{x-5} + 30\int \frac{dx}{x-6}$$

$$= x + 3\log(x-4) - 24\log(x-5) + 30\log(x-6) + C$$