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Concept based notes

CONM

(B.Tech)

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Preface

I am glad to present this book, especially designed to serve the needs of the students. The book has been written keeping in mind the general weakness in understanding the fundamental concepts of the topics. The book is self-explanatory and adopts the "Teach Yourself" style. It is based on question-answer pattern. The language of book is quite easy and understandable based on scientific approach.

Any further improvement in the contents of the book by making corrections, omission and inclusion is keen to be achieved based on suggestions from the readers for which the author shall be obliged.

I acknowledge special thanks to Mr. Rajeev Biyani, *Chairman* & Dr. Sanjay Biyani, *Director* (*Acad.*) Biyani Group of Colleges, who are the backbones and main concept provider and also have been constant source of motivation throughout this Endeavour. They played an active role in coordinating the various stages of this Endeavour and spearheaded the publishing work.

I look forward to receiving valuable suggestions from professors of various educational institutions, other faculty members and students for improvement of the quality of the book. The reader may feel free to send in their comments and suggestions to the under mentioned address.

Author

Unit 1
Matrix

Types of Matrix

Row Matrix

A row matrix is formed by a single row.

Column Matrix

A column matrix is formed by a single column.

Rectangular Matrix

A rectangular matrix is formed by a different number of rows and columns, and its dimension is noted as: **mxn**.

Square Matrix

A square matrix is formed by the same number of rows and columns.

Zero Matrix

In a zero matrix, all the elements are zeros.

Upper Triangular Matrix

In an upper triangular matrix, the elements located below the diagonal are zeros.

Lower Triangular Matrix

In a lower triangular matrix, the elements above the diagonal are zeros.

Diagonal Matrix

In a diagonal matrix, all the elements above and below the diagonal are zeros.

Scalar Matrix

A scalar matrix is a diagonal matrix in which the diagonal elements are equal.

Identity Matrix

An identity matrix is a diagonal matrix in which the diagonal elements are equal to 1.

Transpose Matrix

Given matrix A, the transpose of matrix A is another matrix where the elements in the columns and rows have switched. In other words, the rows become the columns and the columns become the rows.

Regular Matrix

A regular matrix is a square matrix that has an inverse.

Singular Matrix

A singular matrix is a square matrix that has no inverse.

Idempotent Matrix

The matrix A is idempotent if:

$\mathbf{A^2 = A}$.

Involutive Matrix

The matrix A is involutive if:

$\mathbf{A^2 = I}$.

Symmetric Matrix

A symmetric matrix is a square matrix that verifies:

$\mathbf{A = A^t}$.

Antisymmetric Matrix

An antisymmetric matrix is a square matrix that verifies:

$\mathbf{A = -A^t}$.

Orthogonal Matrix

A matrix is orthogonal if it verifies that:

$\mathbf{A \cdot A^t = I}$.

Q.1 Find the inverse of a matrix using gauss elimination:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Ans. Write the augmented matrix as –

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 2 & 2 & 0 & 1 \end{bmatrix}$$

$R_2 \rightarrow R_2 - a_{2,} R_1$
$R_2 \rightarrow R_3 - a_{3,} R_1$
Hence, after the first elimination step, the augmented matrix becomes

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -0 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{bmatrix}$$

$R_1 \rightarrow R_1 - a_{12} R_2$
$R_3 - R_3 - a_{32} R_2$
Hence after the second elimination step, the augmented matrix becomes –

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 2 & 2 & -2 & 0 & 1 \end{bmatrix}$$

$R_1 \rightarrow R_1 - a_{13} R_3$
$R_2 \rightarrow R_2 - a_{23} R_3$

Hence after the third elimination the augmented matrix becomes.

$$\begin{bmatrix} 1 & 1 & 0 & 2/3 & 2/3 & -1/3 \\ 0 & 1 & 0 & 1/3 & -2/3 & 1/3 \\ 0 & 0 & 1 & -2/3 & 1/3 & 1/3 \end{bmatrix}$$

Hence,

$$A^{-1} = \begin{bmatrix} 2/3 & 2/3 & -1/3 \\ 1/3 & -2/3 & 1/3 \\ -2/3 & 1/3 & 1/3 \end{bmatrix}$$

Q.2 Determine the transpose of a matrix

$$\mathbf{A} = \begin{bmatrix} 9 & 4 & 3 & 7 \\ 2 & 4 & 5 & 8 \\ 6 & 7 & 8 & 9 \end{bmatrix}$$

Ans

$$A^T = \begin{bmatrix} 9 & 2 & 6 \\ 4 & 4 & 7 \\ 3 & 5 & 8 \\ 7 & 8 & 9 \end{bmatrix}$$

Find the rank of $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

$R_2 \rightarrow R_2 - R_1$
$R_3 \rightarrow R_3 - 2R_1$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since there are two men- zero rows
∴ Rank A= 2

Unit 2

Gauss Elimination Method

Q.1. Use gauss elimination method to solve :

x + y + z = 7

3x + 3y + 4z = 24

2x + y + 3z = 16

An s.: Since in the first column the largest element is 3 in the second equation, so interchanging the first equation with second equation and making 3 as first pivot.

$$3x + 3y + 4z = 24 \qquad \_\_\_ (1)$$

$$x + y + z = 7 \qquad \_\_\_ (2)$$

$$2x + y + 3z = 16 \qquad \_\_\_ (3)$$

Now eliminating x form equation (2) and equation (3) using equation (1)

-3 × equation (2) + 2 × equation (1), 3 × equation (3) – 2 × equation (1)

we get

$$\begin{array}{r} -3x - 3y - 3z = -21 \\ 3x + 3y + 4z = 24 \\ \hline z = 3 \end{array} \qquad \text{and} \qquad \begin{array}{r} 6x + 3y + 9z = 48 \\ -6x \pm 6y \pm 8z = 48 \\ \hline -3y + z = 0 \end{array}$$

$$= 3y - z = 0$$

$$3x + 3y + 4z = 24 \qquad \_\_\_ (4)$$

$z = 3$ _ _ _ (5)

$3y - z = 0$ _ _ _ (6)

Now since the second row cannot be used as the pivot row since $a_{22} = 0$ so interchanging the equation (5) and (6) we get

$3x + 3y + 4z = 24$ _ _ _ (7)

$3y - z = 0$ _ _ _ (8)

$z = 3$ _ _ _ (9)

Now it is upper triangular matrix system. So by back substitution we obtain.

$\boxed{z = 3}$

From equation (8)

$3y - 3 = 0$

$3y = 3$

$\boxed{y = 1}$

From equation (7)

$3x + 3(1) + 4(3) = 24$

$3x + 3 + 12 = 24$

$3x + 15 = 24$

$3x = 9$

$\boxed{x = 3}$

Hence the solution fo given system of linear equation is

$x = 3 \quad , \quad y = 1 \quad , \quad z = 3$

Q.2. Solve the following system of linear equation by Gauss Elimination Method :

$\mathbf{2x_1 + 4x_2 + x_3 = 3}$

$\mathbf{3x_1 + 2x_2 - 2x_3 = -2}$

$\mathbf{x_1 - x_2 + x_3 = 6}$

Ans.: Since in the first column the largest element is 3 in the second row, so interchanging first equation with second equation and making 3 as first pivot.

$3x_1 + 2x_2 - 2x_3 = -2$ ___ (1)

$2x_1 + 4x_2 + x_3 = 3$ ___ (2)

$x_1 - x_2 + x_3 = 6$ ___ (3)

Eliminating x_1 form equation (2) and equation (3) using equation (1)

-3 × equation (2) + 2 × equation (1) and + 3 × equation (3) – equation (1)

$$\begin{array}{r} -6x_1 - 12x_2 - 3x_3 = -9 \\ 6x_1 + 4x_2 - 4x_3 = -4 \\ \hline -8x_2 - 7x_3 = -13 \end{array} \quad \text{and} \quad \begin{array}{r} 3x_1 - 3x_2 + 3x_3 = 18 \\ 3x_1 + 2x_2 - 2x_3 = -2 \\ \hline -5x_2 + 5x_3 = 20 \end{array}$$

$8x_2 + 7x_3 = 13$ $\qquad x_2 - x_3 = -4$

So the system now becomes :

$3x_1 + 2x_2 - 2x_3 = -2$ ___ (4)

$8x_2 + 7x_3 = 13$ ___ (5)

$x_2 - x_3 = -4$ ___ (6)

Now eliminating x_2 from equation (6) using equation (5) {8 × equation (6) – equation (5)}

$$\begin{array}{r} 8x_2 - 8x_3 = -32 \\ 8x_2 + 7x_3 = -13 \\ \hline -15x_3 = -45 \end{array}$$

$x_3 = 3$

So the system of linear equation is

$3x_1 + 2x_2 - 2x_3 = -2$ ___ (7)

$8x_2 + 7x_3 = 13$ ___ (8)

$x_3 = 3$ ___ (6)

Now it is upper triangular system so by back substitution we obtain

$x_3 = 3$

From equation (8)

$8x_2 + 7(3) = 13$

$8x_2 = 13 - 21$

$8x_2 = - 8$

$x_2 = - 1$

From equation (9)

$3x_1 + 2(-1) - 2(3) = -2$

$3x_1 = -2 + 2 + 6$

$3x_1 = 6$

$x_1 = 2$

∴ Hence the solution of the given system of linear equation is :

$x_1 = 2$, $x_2 = - 1$, $x_3 = 3$

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Gauss-Jordan Elimination Method

Q.1. Solve the following system of equations :

$10x_1 + 2x_2 + x_3 = 9$ _ _ _ (1)

$2x_1 + 20x_2 - 2x_3 = -44$ _ _ _ (2)

$-2x_1 + 3x_2 + 10x_3 = 22$ _ _ _ (3)

Use Gauss Jordan Method.

Ans.: Since in the given system pivoting is not necessary. Eliminating x_1 from equation (2) and equation (3) using equation (1)

5 × equation (2) – equation (1) , 5 × equation (3) + equation (1)

$$\begin{array}{r} 10x_1 - 100x_2 - 10x_3 = -220 \\ -10x_1 + 2x_2 + x_3 = 9 \\ \hline 98x_2 - 11x_3 = -229 \end{array} \quad \text{and} \quad \begin{array}{r} -10x_1 + 15x_2 + 50x_3 = 110 \\ 10x_1 + 2x_2 + x_3 = 9 \\ \hline 17x_2 + 51x_3 = 119 \\ = x_2 + 3x_3 = 7 \end{array}$$

Now the system of equation becomes

$10x_1 + 2x_2 + x_3 = 9$ _ _ _ (4)

$98x_2 - 11x_3 = -229$ _ _ _ (5)

$x_2 + 3x_3 = 7$ _ _ _ (6)

Now eliminating x_2 from equation (4) and (6) using equation (5)

98 × equation (6) – equation (5) , 49 × equation (4) - equation (5)

$$\begin{array}{r} 98x_2 + 294x_3 = 686 \\ -98x_2 \mp 11x_3 = \mp 229 \\ \hline 305x_3 = 915 \\ x_3 = 3 \end{array} \qquad \begin{array}{r} 490x_1 + 98x_2 + 49x_3 = 441 \\ - 98x_2 + -11x_3 = + 9 \\ \hline 490x_1 + 60x_3 = 670 \\ = 49x_1 + 6x_3 = 67 \end{array}$$

Now the system of equation becomes :

$49x_1 + 0 + 6x_3 = 67$ _ _ _ (7)

$98x_2 - 11x_3 = -229$ _ _ _ (8)

$x_3 = 3$ _ _ _ (9)

Hence it reduces to upper triangular system now by back substitution.

$x_3 = 3$

From equation (8)

$98x_2 - 11 \times 3 = -229$

$98x_2 = -229 + 33$

$98x_2 = -196$

$x_2 = -2$

From equation (7)

$49x_1 + 6(3) = 67$

$49x_1 = 67 - 18$

$49x_1 = 49$

$x_1 = 1$

Thus the solution of the given system of linear equation is

$x_1 = 1$, $x_2 = -2$, $x_3 = 3$

Gauss Seidel Method

[This method is also called the method of successive displacement]

Q.1. Solve the following linear equation :

$$2x_1 - x_2 + x_3 = 5$$

$$x_1 + 2x_2 + 3x_3 = 10$$

$$x_1 + 3x_2 - 2x_3 = 7$$

(Use Gauss Seidel Method)

Ans.: Above system of equations can be written as :

$$2x_1 - x_2 + x_3 = 5 \quad \text{--- (1)}$$

$$x_1 + 3x_2 - 2x_3 = 7 \quad \text{--- (2)}$$

$$x_1 + 2x_2 + 3x_3 = 10 \quad \text{--- (3)}$$

Iterative equations are :

$$x_1^{(n+1)} = \frac{1}{2}[5 + x_2^{(n)} - x_3^{(n)}] \quad \text{--- (4)}$$

$$x_2^{(n+1)} = \frac{1}{3}[7 - x_1^{(n+1)} + 2x_3^{(n)}] \quad \text{--- (5)}$$

$$x_3^{(n+1)} = \frac{1}{3}[10 - x_1^{(n+1)} - 2x_2^{(n+1)}] \quad \text{--- (6)}$$

Taking initial approximations as :

$$x_1^{(0)} = 0 \quad ; \quad x_2^{(0)} = 0 \quad \text{and} \quad x_3^{(0)} = 0$$

First approximation is :

$$x_1^{(1)} = \frac{1}{2}[5 + x_2^{(0)} - x_3^{(0)}]$$

$$= \frac{1}{2}[5 + 0 - 0] = \frac{5}{2} = 2.5$$

$$x_2^{(1)} = \frac{1}{3}[7 - x_1^{(1)} + 2x_3^{(0)}]$$

$$= \frac{1}{3}[7 - 2.5 + 2 \times 0] = \frac{1}{3}(4.5) = 1.5$$

$$x_3^{(1)} = \frac{1}{3}[10 - x_1^{(1)} - 2x_2^{(1)}]$$

$$= \frac{1}{3}[10 - 2.5 - 2 \times 1.5] = 1.5$$

Now second approximation :

$$x_1^{(2)} = \frac{1}{2}[5 + x_2^{(1)} - x_3^{(1)}]$$

$$= \frac{1}{2}[5 + (1.5) - 1.5] = 2.5$$

$$x_2^{(2)} = \frac{1}{3}[7 - x_1^{(2)} + 2x_3^{(1)}]$$

$$= \frac{1}{3}[7 - 2.5 + 2(1.5)] = 2.5$$

$$x_3^{(2)} = \frac{1}{3}[10 - x_1^{(2)} - 2x_2^{(2)}]$$

$$= \frac{1}{3}[10 - 2.5 - 2 \times 2.5] = 0.8333$$

$$x_1^{(3)} = \frac{1}{2}[5 + x_2^{(2)} - x_3^{(2)}]$$

$$= \frac{1}{2}[5 + 2.5 - 0.8333] = 3.3333$$

$$x_2^{(3)} = \frac{1}{3}[7 - x_1^{(3)} + 2x_3^{(2)}]$$

$$= \frac{1}{3}[7 - 3.3333 + 2 \times 0.8333] = 1.7777$$

$$x_3^{(3)} = \frac{1}{3}[10 - x_1^{(3)} - 2x_2^{(3)}]$$

$$= \frac{1}{3}[10 - 3.3333 - 2 \times 1.7777] = 1.0371$$

$\therefore \quad x_1^{(3)} = 3.3333 \quad , \quad x_2^{(3)} = 1.7777 \quad , \quad x_3^{(3)} = 1.0371$

$$x_1^{(4)} = \frac{1}{2}[5 + x_2^{(3)} - x_3^{(3)}]$$

$$= \frac{1}{2}[5 + 1.7777 - 1.0371] = 2.8703$$

$x_2^{(4)} = 2.0679$

$x_3^{(4)} = 0.9980$

$\therefore \quad x_1^{(4)} = 2.8703 \quad , \quad x_2^{(4)} = 2.0679 \quad , \quad x_3^{(4)} = 0.9980$

Now $x_1^{(5)} = 3.035$

$x_2^{(5)} = 1.9870$

$x_3^{(5)} = 0.9970$

$x_1^{(6)} = 2.9950$

$x_2^{(6)} = 1.9997$

$x_3^{(6)} = 1.0019$

$x_1^{(7)} = 2.9989$

$x_2^{(7)} = 2.0016$

$x_3^{(7)} = 0.9993$

$x_1^{(8)} = 3.0011$

$x_2^{(8)} = 1.9991$

$x_3^{(8)} = 1.0002$

$x_1^{(9)} = 2.9994$

$x_2^{(9)} = 2.0003$

$x_3^{(9)} = 1$

$x_1^{(10)} = 3.0001$

$x_2^{(10)} = 1.9999$

$x_3^{(10)} = 1$

$x_1^{(11)} = 2.9999$

$x_2^{(11)} = 2$

$x_3^{(11)} = 1$

$x_1^{(12)} = 3$

$x_2^{(12)} = 2$

$x_3^{(12)} = 1$

$x_1^{(13)} = 3$

$x_2^{(13)} = 2$

$x_3^{(13)} = 1$

Hence the solution of the given system of linear equation is :

$x_1 = 3$, $x_2 = 2$, $x_3 = 1$

□ □ □

Unit 3

Bisection Method

Q.1. Find real root of the equation $x^3 - 5x + 3$ upto three decimal digits.

Ans.: Here $f(x) = x^3 - 5x + 3$

$f(0) = 0 - 0 + 3 = 3 \; f(x_0)$ (say)

$f(1) = 1 - 5 + 3 = -1 = f(x_1)$ (say)

Since $f(x_0), f(x_1) < 0$ so the root of the given equation lies between 0 and 1

So, $x_2 = \dfrac{x_0 + x_1}{2} = \dfrac{0+1}{2} = 0.5$

Now, $f(x_2) = f(0.5)$

$= (0.5)^3 - 5\,(0.5) + 3$

$= 0.125 - 2.5 + 3$

$= 0.625$ (which is positive)

$\because \quad f(x_1).f(x_2) < 0$

So, $x_3 = \dfrac{x_1 + x_2}{2} = \dfrac{1 + 0.5}{2} = 0.75$

Now, $f(x_3) = f(0.75)$

$= (0.75)^3 - 5\,(0.75) + 3$

$= 0.4218 - 3.75 + 3$

$= -0.328$ (which is negative)

$\because \quad f(x_2).f(x_3) < 0$

So, $x_4 = \frac{x_2 + x_3}{2} = \frac{0.5 + 0.75}{2} = 0.625$

Now, $f(x_4) = f(0.625)$

$= (0.625)^3 - 5\,(0.625) + 3$

$= 0.244 - 3.125 + 3$

$= 0.119$ (which is positive)

$\because \quad f(x_3).f(x_4) < 0$

So, $x_5 = \frac{x_3 + x_4}{2} = \frac{0.75 + 0.625}{2} = 0.687$

Now, $f(x_5) = f(0.687)$

$= (0.687)^3 - 5\,(0.687) + 3$

$= -\,0.1108$ (which is negative)

$\because \quad f(x_4).f(x_5) < 0$

So, $x_6 = \frac{x_4 + x_5}{2} = \frac{0.625 + 0.687}{2} = 0.656$

Now, $f(x_6) = f(0.656)$

$= (0.656)^3 - 5\,(0.656) + 3$

$= 0.0023$ (which is positive)

$\because \quad f(x_5).f(x_6) < 0$

So, $x_7 = \frac{x_5 + x_6}{2} = \frac{0.687 + 0.656}{2} = 0.671$

Now, $f(x_7) = f(0.671)$

$= (0.671)^3 - 5\,(0.671) + 3$

$= -\,0.0528$ (which is negative)

$\because \quad f(x_6).f(x_7) < 0$

So, $x_8 = \frac{x_6 + x_7}{2} = \frac{0.656 + 0.671}{2} = 0.663$

Now, $f(x_8) = f(0.663)$

$= (0.663)^3 – 5\ (0.663) + 3$

$= 0.2920 – 3.315 + 3$

$= – 0.023$ (which is negative)

$\because \quad f(x_6).f(x_8) < 0$

So, $x_9 = \dfrac{x_6 + x_8}{2} = \dfrac{0.656 + 0.663}{2} = 0.659$

Now, $f(x_9) = f(0.659)$

$= (0.659)^3 – 5\ (0.659) + 3$

$= – 0.0089$ (which is negative)

$\because \quad f(x_6).f(x_9) < 0$

So, $x_{10} = \dfrac{x_6 + x_9}{2} = \dfrac{0.656 + 0.659}{2} = 0.657$

Now, $f(x_{10}) = f(0.657)$

$= (0.657)^3 – 5\ (0.657) + 3$

$= – 0.00140$ (which is negative)

$\because \quad f(x_6).f(x_{10}) < 0$

So, $x_{11} = \dfrac{x_6 + x_{10}}{2} = \dfrac{0.656 + 0.657}{2} = 0.656$

Now, $f(x_{11}) = f(0.656)$

$= (0.656)^3 – 5\ (0.656) + 3$

$= 0.2823 – 3.28 + 3$

$= 0.00230$ (which is positive)

$\because \quad f(x_{11}).f(x_{10}) < 0$

So, $x_{12} = \dfrac{x_{10} + x_{11}}{2} = \dfrac{0.657 + 0.656}{2} = 0.656$

Since x_{11} and x_{12} both same value. Therefore if we continue this process we will get same value of x so the value of x is 0.565 which is required result.

Q.2. Find real root of the equation cos x - xe^x = 0 correct upto four decimal places.

Ans.: Since, $f(x) = \cos x - xe^x$

So, $f(0) = \cos 0 - 0e^0 = 1$ (which is positive)

And $f(1) = \cos 1 - 1e^1 = -2.1779$ (which is negative)

$\because$ $f(0).f(1) < 0$

Hence the root of are given equation lies between 0 and 1.

let $f(0) = f(x_0)$ and $f(1) = f(x_1)$

So, $x_2 = \frac{x_0 + x_1}{2} = \frac{0 + 1}{2} = 0.5$

Now, $f(x_2) = f(0.5)$

$f(0.5) = \cos(0.5) - (0.5)e^{0.5}$

$= 0.05322$ (which is positive)

$\because$ $f(x_1).f(x_2) < 0$

So, $x_3 = \frac{x_1 + x_2}{2} = \frac{1 + 0.5}{2} = \frac{1.5}{2} = 0.75$

Now, $f(x_3) = f(0.75)$

$= \cos(0.75) - (0.75)e^{0.75}$

$= - 0.856$ (which is negative)

$\because$ $f(x_2).f(x_3) < 0$

So, $x_4 = \frac{x_2 + x_3}{2} = \frac{0.5 + 0.75}{2} = 0.625$

$f(x_4) = f(0.625)$

$= \cos(0.625) - (0.625)e^{(0.625)}$

$= - 0.356$ (which is negative)

$\because$ $f(x_2).f(x_4) < 0$

So, $x_5 = \frac{x_2 + x_4}{2} = \frac{0.5 + 0.625}{2} = 0.5625$

Now, $f(x_3) = f(0.5625)$

$= \cos(0.5625) - 0.5625e^{0.5625}$

$= -0.14129$ (which is negative)

$\because \quad f(x_2).f(x_5) < 0$

So, $x_6 = \frac{x_2 + x_5}{2} = \frac{0.5 + 0.5625}{2} = 0.5312$

Now, $f(x_6) = f(0.5312)$

$= \cos(0.5312) - (0.5312)e^{0.5312}$

$= -0.0415$ (which is negative)

$\because \quad f(x_2).f(x_6) < 0$

So, $x_7 = \frac{x_2 + x_6}{2} = \frac{0.5 + 0.5312}{2} = 0.5156$

Now, $f(x_7) = f(0.5156)$

$= \cos(0.5156) - (0.5156)e^{0.5156}$

$= 0.006551$ (which is positive)

$\because \quad f(x_6).f(x_7) < 0$

So, $x_8 = \frac{x_6 + x_7}{2} = \frac{0.513 + 0.515}{2} = 0.523$

Now, $f(x_8) = f(0.523)$

$= \cos(0.523) - (0.523)e^{0.523}$

$= -0.01724$ (which is negative)

$\because \quad f(x_7).f(x_8) < 0$

So, $(x_9) = \frac{x_7 + x_8}{2} = \frac{0.515 + 0.523}{2} = 0.519$

Now, $f(x_9) = f(0.519)$

$= \cos(0.519) - (0.519)e^{0.519}$

$= -0.00531$ (which is negative)

$\because \quad f(x_7).f(x_9) < 0$

So, $(x_{10}) = \dfrac{x_7 + x_9}{2} = \dfrac{0.515 + 0.519}{2} = 0.5175$

Now, $f(x_{10}) = f(0.5175)$

$= \cos(0.5175) - (0.5175)e^{0.5175}$

$= 0.0006307$ (which is positive)

$\because \quad f(x_9).f(x_{10}) < 0$

So, $x_{11} = \dfrac{x_9 + x_{10}}{2} = \dfrac{0.5195 + 0.5175}{2} = 0.5185$

Now, $f(x_{11}) = f(0.5185)$

$= \cos(0.5185) - (0.5185)e^{0.5185}$

$= -\ 0.002260$ (which is negative)

$\because \quad f(x_{10}).f(x_{11}) < 0$

So, $x_{12} = \dfrac{x_{10} + x_{11}}{2} = \dfrac{0.5175 + 0.5185}{2} = 0.5180$

Hence the root of the given equation upto 3 decimal places is x = 0.518

Thus the root of the given equation is x = 0.518

□ □ □

Regula Falsi Method

Q.1. Find the real root of the equation x $\log_{10}$ x – 1.2 = 0 correct upto four decimal places.

Ans.: Given $f(x) = x \log_{10} x - 1.2$ _ _ _ (1)

In this method following formula is used -

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})\ f(x_n)}{(f(x_n) - f(x_{n-1}))} \quad \text{\_ \_ \_ (2)}$$

Taking x = 1 in eq.(1)

$f(1) = 1.\ \log_{10} 1 - 1.2$

$= -2$ (which is negative)

Taking x = 2 in eq.(1)

$f(2) = 2.\ \log_{10} 2 - 1.2$

$= -0.5979$ (which is negative)

Taking x = 3 in eq.(1)

$f(3) = 3.\ \log_{10} 3 - 1.2$

$= 0.2313$ (which is positive)

$\because \quad f(2).f(3) < 0$

So the root of the given equation lies between 2 and 3.

let $x_1 = 2$ and $x_2 = 3$

$\therefore \quad f(x_1) = f(2) = -0.5979$

And $f(x_2) = f(3) = 0.2313$

Now we want to find x_3 so using eq.(2)

$$x_3 = x_2 - \frac{(x_2 - x_1)\ f(x_2)}{f(x_2) - f(x_1)}$$

$$= 3 - \frac{(3-2)\ \times\ (0.2313)}{0.2313 - (-0.5979)}$$

$$= 3 - \frac{0.2313}{0.8292}$$

$$= 3 - 0.2789\ = 2.7211$$

$$f(x_3) = f(2.7211)$$

$$= 2.7211 \log_{10} 2.7211 - 1.2$$

$$= -0.01701 \text{ (which is negative)}$$

$\because \quad f(x_2).f(x_3) < 0$

Now to find x_4 using equation (2)

$$x_4 = x_3 - \frac{(x_3 - x_2)\ f(x_3)}{f(x_3) - f(x_2)}$$

$$= 2.7211 - \frac{(2.7211 - 3) \times (-0.0170)}{(-0.0170 - 0.2313)}$$

$$= 2.7211 - \frac{0.004743}{0.2483}$$

$$= 2.7211 + 0.01910\ = 2.7402$$

Now

$$f(x_4) = f(2.7402)$$

$$= 2.7402 \log_{10} 2.7402 - 1.2$$

$$= -0.0003890 \text{ (which is negative)}$$

$\because \quad f(x_2).f(x_4) < 0$

Now to find x_5 using equation (2)

$$x_5 = x_4 - \frac{(x_4 - x_2)\ f(x_4)}{[f(x_4) - f(x_2)]}$$

$$= 2.7402 - \frac{(2.7402 - 3)}{(-0.0004762 - 0.2313)} \times (-0.0004762)$$

$$= 2.7402 + \frac{(-0.2598)(-0.0004762)}{0.2317}$$

$$= 2.7402 + \frac{(0.0001237)}{0.2317}$$

$$= 2.7402 + 0.0005341 = 2.7406$$

$f(x_5) = f(2.7406)$

$= 2.7406 \log_{10} 2.7406 - 1.2$

$= -\ 0.0000402$ (which is negative)

$\because \quad f(x_2).f(x_5) < 0$

To find x_6 using equation (2)

$$x_6 = x_5 - \frac{(x_5 - x_2)\ f(x_5)}{f(x_5) - f(x_2)}$$

$$= 2.7406 + \frac{(2.7406 - 3) \times (-0.000040)}{(-0.00004) - (0.2313)}$$

$$= 2.7406 + 0.000010 = 2.7406$$

$\therefore$ The approximate root of the given equation is 2.7406 which is correct upto four decimals.

Q.2. Find the real root of the equation $x^3 - 2x - 5 = 0$ correct upto four decimal places.

Ans.: Given equation is

$f(x) = x^3 - 2x - 5$ _ _ _ (1)

In this method following formula is used :-

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})\ f(x_n)}{[f(x_n) - f(x_{n-1})]} \qquad \text{---} (2)$$

Taking x = 1 in equation (1)

$f(1) = 1 - 2 - 5 = -6$ (which is negative)

Taking x = 2 in equation (1)

$f(2) = 8 - 4 - 5 = -1$ (which is negative)

Taking x = 3

$f(3) = 27 - 6 - 5 = 16$ (which is positive)

Since $f(2).f(3) < 0$

So the root of the given equation lies between 2 and 3.

Let $x_1 = 2$ and $x_2 = 3$

$f(x_1) = f(2) = -1$

and $f(x_2) = f(3) = 16$

Now to find x_3 using equation (2)

$$x_3 = x_2 - \frac{(x_2 - x_1)}{f(x_2) - f(x_1)} f(x_2)$$

$$= 3 - \frac{(3 - 2)}{16 + 1} \times 16$$

$$= 3 - \frac{16}{17} = 2.0588$$

$f(x_3) = (2.0558)^3 - 2\ (2.0588) - 5$

$= 8.7265 - 4.1176 - 5$

$= -0.3911$ (which is negative)

$\therefore \quad f(x_2).f(x_3) < 0$

Now to find x_4 using equation (2)

$$x_4 = x_3 - \frac{(x_3 - x_2)}{[f(x_3) - f(x_2)]} \times f(x_3)$$

$$= 2.0588 - \frac{(2.0588 - 3)}{-0.3911 - 16} \times (-0.3911)$$

$$= 2.0588 + \frac{(-0.9412)\times(-0.3911)}{16.3911} = 2.0812$$

$\therefore$ $f(x_4) = 9.0144 - 4.1624 - 5$

$= -0.148$ (which is negative)

So $f(x_2) \,.\, f(x_4) < 0$

Now using equation (2) to find x_5

$$x_5 = x_4 - \frac{(x_4 - x_2)}{[f(x_4) - f(x_2)]} \times f(x_4)$$

$$= 2.0812 - \frac{(2.0812 - 3)}{(-0.148 - 16)} \times (-0.148)$$

$$= 2.0812 + \frac{(-0.9188)\times(-0.148)}{16.148}$$

$$= 2.0812 + 8.4210 \text{ x } \frac{(x_5 - x_2) \times f(x_5)}{f(x_5) - f(x_2)} 10^{-3}$$

$= 2.0896$

$\therefore$ $f(x_5) = 9.1240 - 4.1792 - 5$

$= -0.0552$ (which is negative)

$f(x_2).f(x_5) < 0$

Now using equation (2) to find x_6

$$x_6 = x_5 - \frac{(x_5 - x_2) \times f(x_5)}{f(x_5) - f(x_2)}$$

$$= 2.0896 - \frac{(2.0896 - 3)}{(-0.0552 - 16)} \times (-0.0552)$$

$$= 2.0896 + \frac{(0.05025)}{16.0552}$$

$$= 2.0927$$

$\therefore$ $f(x_6) = 9.1647 – 4.1854 – 5$

$= – 0.0207$ (which is negative)

So $f(x_2).f(x_6) < 0$

Now using equation (2) to find x_7

$$x_7 = x_6 – \frac{(x_6 - x_2)}{f(x_6) - f(x_2)} \times f(x_6)$$

$$= 2.0927 – \frac{(2.0927 - 3)}{(-0.0207 - 16)} \times (-0.0207)$$

$$= 2.0927 + \frac{(-0.9073)\ (-0.0207)}{16.0207}$$

$$= 2.0927 + 1.1722 \text{ x } 10^{-3}$$

$$= 2.0938$$

Now $f(x_7) = 9.1792 – 4.1876 – 5$

$= – 0.0084$ (which is negative)

So $f(x_2).f(x_7) < 0$

Now using equation (2) to find x_8

$$x_8 = x_7 – \frac{(x_7 - x_2)}{f(x_7) - f(x_2)} \times f(x_7)$$

$$= 2.0938 – \frac{(2.0938 - 3)}{(-0.0084 - 16)} \times (-0.0084)$$

$$= 2.0938 + \frac{(-0.9062)\ (-0.0084)}{16.0084}$$

$$= 2.0938 + 4.755 \text{ x } 10^{-4}$$

$$= 2.09427$$

∴ $f(x_8) = 9.1853 – 4.18854 – 5$

$= – 0.00324$ (which is negative)

So $f(x_2).f(x_8) < 0$

Now using equation (2) to find x_9

$$x_9 = x_8 - \frac{(x_8 - x_2)}{f(x_8) - f(x_2)} \times f(x_8)$$

$$= 2.09427 - \frac{(2.09427 - 3)}{(-0.00324 - 16)} \times (-0.00324)$$

$$= 2.09427 - \frac{(-0.90573)(-0.00324)}{16.00324}$$

$$= 2.0944$$

∴ The real root of the given equation is 2.094 which is correct upto three decimals.

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Secant Method

Note : In this method following formula is used to find root –

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1}) f(x_n)}{f(x_n) - f(x_{n-1})} \quad \text{\_ \_ \_ (1)}$$

Q.1. Find the root of the equation $x^3 - 5x^2 - 17x + 20$ [use Secant Method] correct upto four decimals.

Ans.: Given $f(x) = x^3 - 5x^2 - 17x + 20$ _ _ _ (2)

Taking x = 0 in equation (1)

$f(0) = 20$

Now taking x = 1

$f(1) = 1 - 5 - 17 + 20$

$= -1$

Since $f(0) = 20$ (positive) and $f(1) = -1$ (which is negative) so the root of the given equation lies between 0 and 1.

Let $x_1 = 0$ and $x_2 = 1$

$\therefore \quad f(x_1) = 20$ and $f(x_2) = -1$

using equation (1) to find x_3

$$x_3 = x_2 - \frac{(x_2 - x_1)}{f(x_2) - f(x_1)} \times f(x_2)$$

$$= 1 - \frac{(1 - 0)}{(-1) - 20} \times (-1)$$

$$= 1 + \frac{(1)}{(-21)} = 1 - \frac{1}{21}$$

= 0.9523

$\therefore$ $f(x_3)$ = $f(0.9523)$

= $(0.9523)^3 - 5\,(0.9523)^2 - 17\,(0.9523) + 20$

= 0.8636 – 4.5343 – 16.1891 + 20

= 0.1402 (which is positive)

Using equation (1) to find x_4

$$x_4 = x_3 - \frac{(x_3 - x_2)}{f(x_3) - f(x_2)} \times f(x_3)$$

$$= 0.9523 - \frac{(0.9523 - 1)}{[0.1402 - (-1)]} \times 0.1402$$

$$= 0.9523 - \frac{(-0.0477)\,(0.1402)}{(1.1402)}$$

= 0.9523 + 0.005865 = 0.9581

$f(x_4) = (0.9581)^3 - 5\,(0.9581)^2 - 17\,(0.9581) + 20$

= 0.8794 – 4.5897 – 16.2877 + 20

= 0.0020 (which is positive)

$$x_5 = x_4 - \frac{(x_4 - x_3)}{f(x_4) - f(x_3)} \times f(x_4)$$

$$= 0.9581 - \frac{(0.9581 - 0.9523)}{(0.0020) - (0.1402)} \times 0.0020$$

= 0.9581

Hence the root of the given equation is 0.9581 which is correct upto four decimal.

Q.2. Given that one of the root of the non-linear equation cos x – xe^x = 0 lies between 0.5 and 1.0 find the root correct upto three decimal places, by Secant Method.

Ans.: Given equation is $f(x) = \cos x - xe^x$

And $x_1 = 0.5$ and $x_2 = 1.0$

$f(x_1) = \cos(0.5) - (0.5)\, e^{0.5}$

$= 0.87758 - 0.82436$

$= 0.05321$

Now $f(x_2) = \cos(1) - (1)\, e^{1}$

$= 0.54030 - 2.71828$

$= -2.1780$

Now to calculate x_3 using equation (1)

$$x_3 = x_2 - \frac{(x_2 - x_1)}{f(x_2) - f(x_1)} \times f(x_2)$$

$$= 1 - \frac{(1 - 0.5)}{(-2.1780 - 0.05321)} \times (-2.1780)$$

$$= 1 - \frac{(0.5)\,(2.1780)}{2.23121}$$

$= 1 - 0.48807$

$= +0.51192$

$\therefore$ $f(x_3) = f(0.51192)$

$= \cos(0.51192) - (0.51192)e^{0.51192}$

$= 0.87150 - 0.85413$

$= 0.01767$

Now for calculating x_4 using equation (1)

$$x_4 = x_3 - \frac{(x_3 - x_2)}{f(x_3) - f(x_2)} \times f(x_3)$$

$$= 0.51192 - \frac{(0.51192 - 1)}{(0.1767) - (-2.1780)} \times 0.01767$$

$$= 0.51192 - \frac{(-0.48808)\,(0.01767)}{2.19567}$$

$$= 0.51192 + \frac{0.0086243}{2.19567}$$

$$= 0.51192 + 0.003927$$

$$= 0.51584$$

$\therefore \quad f(x_4) = \cos(0.51584) - (0.51584)e^{0.51584}$

$$= 0.86987 - 0.86405$$

$= 0.005814$ (which is positive)

Now for calculating x_5 using equation (1)

$$x_5 = x_4 - \frac{(x_4 - x_3)}{f(x_4) - f(x_3)} \times f(x_4)$$

$$= 0.51584 - \frac{(0.51584 - 0.51192)}{(0.005814 - 0.01767)} \times 0.005814$$

$$= 0.51584 - \frac{0.00392}{(-0.01185)} \times (0.005814)$$

$$= 0.51584 + 0.001923$$

$$= 0.51776$$

$$= 0.5178$$

Now $f(x_5) = \cos(0.5178) - (0.5178)e^{0.5178}$

$$= 0.8689 - 0.8690$$

$$= -0.00001$$

$= -0.0000$ (upto four decimals)

Hence the root of the given equation is x = 0.5178 (which is correct upto four decimal places)

This process cannot be proceed further because $f(x_5)$ vanishes.

Newton Raphson Method

Hint : Formula uses in this method is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Q.1. Find the root of the equation $x^2 - 5x + 2 = 0$ correct upto 5 decimal places. (use Newton Raphson Method.)

Ans.: : Given $f(x) = x^2 - 5x + 2 = 0$

Taking x = 0

$f(0) = 2$ (which is positive)

Taking x = 1

$f(1) = 1 - 5 + 2 = -2$ (which is negative)

$f(0) . f(1) < 0$

∴ The root of the given equation lies between 0 and 1

Taking initial approximation as

$x_1 = \frac{0+1}{2} = 0.5$

$f(x) = x^2 - 5x + 2$

$f'(x) = 2x - 5$

Since $x_1 = 0.5$

$f(x_1) = (0.5)^2 - 5(0.5) + 2$

$= 0.25 - 2.5 + 2$

$= -0.25$

$f'(x_1) = 2(0.5) - 5$

$= 1 - 5$

$= -4$

Now finding x_2

$$x_2 = 0.5 - \frac{(-0.25)}{-4}$$

$$= 0.5 - \frac{0.25}{4}$$

$$= 0.4375$$

$$f(x_2) = (0.4375)^2 - 5(0.4375) + 2$$

$$= 0.19140 - 2.1875 + 2$$

$$= 0.003906$$

$$f'(x_2) = 2(0.4375) - 5$$

$$= -4.125$$

Now finding x_3

$$x_3 = x_2 - \frac{f(x_2)}{f^1(x_2)}$$

$$= 0.4375 - \frac{0.003906}{(-4.125)}$$

$$= 0.4375 + 0.0009469$$

$$= 0.43844$$

$$f(x_3) = (0.43844)^2 - 5(0.43844) + 2$$

$$= 0.19222 - 2.1922 + 2$$

$$= 0.00002$$

$$f'(x_3) = 2 \times (0.43844) - 5$$

$$= -4.12312$$

$$x_4 = x_3 - \frac{f(x_3)}{f^1(x_3)}$$

$$= 0.43844 - \frac{0.00002}{(-4.12312)}$$

$= 0.43844 + 0.00000485$

$= 0.43844$

Hence the root of the given equation is 0.43844 which is correct upto five decimal places.

Q.2. Apply Newton Raphson Method to find the root of the equation 3x – cos x – 1 = 0 correct the result upto five decimal places.

Ans.: Given equation is

$f(x) = 3x - \cos x - 1$

Taking x = 0

$f(0) = 3(0) - \cos 0 - 1$

$= -2$

Now taking x = 1

$f(1) = 3(1) - \cos(1) - 1$

$= 3 - 0.5403 - 1$

$= 1.4597$

Taking initial approximation as

$$x_1 = \frac{0+1}{2} = 0.5$$

$f(x) = 3x - \cos x - 1$

$f'(x) = 3 + \sin x$

At $x_1 = 0.5$

$f(x_1) = 3(0.5) - \cos(0.5) - 1$

$= 1.5 - 0.8775 - 1$

$= -0.37758$

$f'(x_1) = 3 - \sin(0.5)$

$= 3.47942$

Now to find x_2 using following formula

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.5 - \frac{(-0.37758)}{(3.47942)}$$

$$= 0.5 + 0.10851$$

$$= 0.60852$$

$$f(x_2) = 3\,(0.60852) - \cos\,(0.60852) - 1$$

$$= 1.82556 - 0.820494 - 1$$

$$= 0.005066$$

$$f'(x_2) = 3 + \sin\,(0.60852)$$

$$= 3.57165$$

Now finding x_3

$$x_3 = 0.60852 - \frac{(0.005066)}{(3.57165)}$$

$$= 0.60852 - 0.0014183$$

$$= 0.60710$$

$$f(x_3) = 3\,(0.60710) - \cos\,(0.60710) - 1$$

$$= 1.8213 - 0.821305884 - 1$$

$$= -0.00000588$$

$$f'(x_3) = 3 + \sin\,(0.60710)$$

$$= 3 + 0.57048$$

$$= 3.5704$$

Now to find x_4 using following formula

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= 0.60710 - \frac{(-0.00000588)}{3.5704}$$

$= 0.60710 + 0.00000164$

$= 0.60710$

Which is same as x_3

Hence the root of the given equation is $x = 0.60710$ which is correct upto five decimal places.

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Unit 4

Numerical Solution for Differential Equations [Euler's Method]

Q.1. Use Euler's Method to determine an approximate value of y at x = 0.2 from initial value problem $\frac{dx}{dy} = 1 - x + 4y$ y(0) = 1 taking the step size h = 0.1.

Ans.: Here h = 0.1, n = 2, x0 = 0, y0 = 1

Given $\frac{dx}{dy} = 1 - x + 4y$

Hence y1 = y0 + hf(x0, y0)

= 1 + 0.1 [1 – x0 + 4y0]

= 1 + 0.1 [1 – 0 + 4 × 1]

= 1 + 0.1 [1 + 4]

= 1 + 0.5 × 5

= 1.5

Similarly y2 = y1 + hf(x0 + h, y1)

= 1.5 + 0.1[1 – 0.1 + 4 × 1.5]

= 2.19

Q.2. Using Euler's Method with step-size 0.1 find the value of y(0.5) from the following differential equation $\frac{dx}{dy}$ = x2 + y2 , y(0) = 0

Ans.: Here h = 0.1, n = 5, x0 = 0, y0 = 0 and f(x, y) = x2 + y2

Hence y1 = y0 + hf(x0, y0)

= 0 + (0.1) [02 + 02]

= 0

Similarly y2 = y1 + hf(x0 + h, y1)

= 0 + (0.1) [(0.1)2 +02]

= (0.1)3

= 0.001

y3 = y2 + hf[x0 + 2h, y2]

= 0.001 + (0.1) [(0.2)2 +(0.001)2]

= 0.001 + 0.1 [0.04 + 0.00001]

= 0.001 + 0.1 [0.040001]

= 0.005

y4 = y3 + hf[x0 + 3h, y3]

= 0.005 + (0.1) [(0.3)2 +(0.005)2]

= 0.005 + (0.1) [0.09 + 0.000025]

= 0.014

y5 = y4 + hf[x0 + 4h, y4]

= 0.014 + (0.1) [(0.4)2 +(0.014)2]

= 0.014 + (0.1) [0.16 + 0.00196]

= 0.031

Hence the required solution is 0.031

□ □ □

Numerical Solution for Differential Equations [Euler's Modified Method]

Q.1. Using Euler's modified method, obtain a solution of the equation $\frac{dy}{dx} = x + |\sqrt{y}|$ with initial conditions y = 1 at x = 0 for the range $0 \le x \le 0.6$ in the step of 0.2. Correct upto four place of decimals.

Ans.: Here $f(x, y) = x + |\sqrt{y}|$

$x_0 = 0$, $y_0 = 1$, $h = 0.2$ and $x_n = x_0 + nh$

(i) At x = 0.2

First approximate value of y_1

$y_1^{(1)} = y_0 + hf(x_0, y_0)$

$= 1 + (0.2)[0 + 1]$

$= 1.2$

Second approximate value of y_1

$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$

$= 1 + \frac{0.2}{2}[(0 + 1) + \{0.2 + \sqrt{1.2}\}]$

$= 1.2295$

Third approximate value of y_1

$y_1^{(3)} = y_0 + \frac{h}{2}\{f(x_0, y_0) + f(x_1, y_1^{(2)})\}$

$= 1 + \frac{0.2}{2}\ [(0 + 1) + \{0.2 + \sqrt{1.2295}\ \}]$

$= 1 + 0.1\ [1 + 1.30882821]$

$= 1.2309$

Fourth approximate value of y1

$y1(4) = y0 + \frac{h}{2}\ \{f(x0, y0) + f(x1, y1(3))\}$

$= 1 + \frac{0.2}{2}\ [(0 + 1) + (0.2 + \sqrt{1.2309})]$

$= 1 + 0.1\ [1 + 1.30945]$

$= 1.2309$

Since the value of y1(3) and y1(4) is same

Hence at x1 = 0.2, y1 = 1.2309

(ii) At x = 0.4

First approximate value of y2

$y2(1) = y1 + hf(x1, y1)$

$= 1.2309 + (0.2)\ \{0.2 + \sqrt{1.2309}\}$

$= 1.4927$

Second approximate value of y2

$y2(2) = y1 + \frac{h}{2}\ [f(x1, y1) + f(x2, y2(1))]$

$= 1.2309 + \frac{0.2}{2}\ [(0.2 + \sqrt{1.2309}) + (0.4 + \sqrt{1.4927}\ )]$

$= 1.2309 + 0.1\ [1.309459328 + (1.621761024]$

$= 1.5240$

Third approximate value of y2

$y2(3) = y1 + \frac{h}{2}\ [f(x1, y1) + f(x2, y2(2))]$

$$= 1.2309 + \frac{0.2}{2}\ [(1.309459328 + (0.4 + \sqrt{1.5240})]$$

= 1.2309 + 0.1 [1.309459328 + 1.634503949]

= 1.5253

Fourth approximate value of y2

$$y2(4) = y1 + \frac{h}{2}\ \{f(x1, y1) + f(x2, y2(3))\}$$

$$= 1.2309 + \frac{0.2}{2}\ [(0.2 + \sqrt{1.2309}) + (0.4 + \sqrt{1.5253})]$$

= 1.2309 + 0.1 {1.309459328 + 1.635030364]

= 1.5253

Hence at x = 0.4, y2 = 1.5253

(ii) At x = 0.6

First approximate value of y3

y3(1) = y2 + hf(x2, y2)

$$= 1.5253 + 0.2\ [0.4 + \sqrt{1.5253}]$$

= 1.8523

Second approximate value of y3

$$y3(2) = y2 + \frac{h}{2}\ \{f(x2, y2) + f(x3, y3(1))\}$$

$$= 1.5253 + \frac{0.2}{2}\ [(0.4 + \sqrt{1.5253}) + (0.6 + \sqrt{1.8523})]$$

= 1.8849

Third approximate value of y3

$$y3(3) = y2 + \frac{h}{2}\ \{f(x2, y2) + f(x3, y3(2))\}$$

$$= 1.5253 + \frac{0.2}{2}\left[(0.4 + \sqrt{1.5253}) + (0.6 + \sqrt{1.8849})\right]$$

$$= 1.8851$$

Fourth approximate value of y3

$$y3(4) = y2 + \frac{h}{2}\{f(x2, y2) + f(x3, y3(3))\}$$

$$= 1.5253 + \frac{0.2}{2}\left[(0.4 + \sqrt{1.5253}) + (0.6 + \sqrt{1.8851})\right]$$

$$= 1.8851$$

Hence at x = 0.6, y3 = 1.8851

□ □ □

Numerical Solution for Differential Equations [Runge – Kutta Method]

Q.1. Using Runge – Kutta method find an approximate value of y for x = 0.2 in step of 0.1 if $\frac{dy}{dx} = x + y^2$ given y = 1 when x = 0

Ans.: Here $f(x, y) = x + y2$, x0 = 0, y0 = 1 and h = 0.1

$K1 = hf(x0, y0) = 0.1[0 + 1]$

$= 0.1$ ………. _ _ _ (1)

$K2 = hf(x0 + \frac{1}{2} h, y0 + \frac{1}{2} K1)$

$= 0.1\left[\left(0+\frac{1}{2}(0.1)\right)+\left(1+\frac{1}{2}\times 0.1152\right)^2\right]$

$= 0.1152$ _ _ _ (2)

$K3 = hf(x0 + \frac{1}{2} h, y0 + \frac{1}{2} K2)$

$= 0.1\left[\left(0+\frac{1}{2}(0.1)\right)+\left\{1+\left(\frac{1}{2}\times 0.1152\right)\right\}^2\right]$

$= 0.1168$ _ _ _ (3)

$K4 = hf(x0 + h, y0 + K3)$

$= 0.1\left[\ 0+0.1\ +\ 1+0.1168\ ^{2}\right]$

$= 0.1347$ _ _ _ (4)

and

$K = \frac{1}{6}(K1 + 2K2 + 2K3 + K4)$

$= \frac{1}{6}\left(0.1 + 2(0.1152) + 2(0.1168) + 0.1347\right)$ {using equation (1), (2), (3) and (4)}

$= 0.1165$

Hence y1 = y0 + K = 1 + 0.1165

= 1.1165 _ _ _ (5)

Again x1 = x0 + h = 0.1, y1 = 1.1165, h = 0.1

Now

$K1 = hf(x1, y1)$

$= 0.1\left[0.1 + (1.1165)^2\right]$

$= 0.1347$ _ _ _ (6)

$K2 = hf\left[x_1 + \frac{1}{2}h,\ y_1 + \frac{1}{2}K_1\right]$

$= 0.1\left[\left\{0.1 + \frac{1}{2}(0.1)\right\} + \left\{1.1165 + \frac{1}{2}(0.1347)\right\}^2\right]$

$= 0.1551$ _ _ _ (7)

$K3 = hf\left[x_1 + \frac{1}{2}h,\ y_1 + \frac{1}{2}K_2\right]$

$= 0.1\left[\left\{0.1 + \frac{1}{2}(0.1)\right\} + \left\{1.1165 + \frac{1}{2}(0.1551)\right\}^2\right]$

$= 0.1576$ _ _ _ (8)

$K4 = hf\left(x_1 + h,\ y_1 + K_3\right)$

$= (0.1)\left[\left(0.1 + 0.1\right) + \left(1.1165 + 0.1576\right)^2\right]$

$= 0.1823$ _ _ _ (9)

and

$$K = \frac{1}{6}(K1 + 2K2 + 2K3 + K4)$$

$$= \frac{1}{6}\ 0.1347 + 2(0.1551) + 2(0.1576) + 0.1823$$ {using equation (6), (7), (8) and (9)}

= 0.1570

Hence

y(0.2) = y2 = y1 + K

= 1.1165 + 0.1570

= 1.2735

which is required solution.

Q.2. Use Runge-Kutta method to solve y' = x y for x = 1.4. Initially x = 1, y = 2 (tale h = 0.2).

[BCA Part II, 2007]

Ans.: (i) Here f(x, y) = xy, x0 = 1, y0 = 2, h = 0.2

K1 = hf(x0, y0)

= 0.2[1 × 2]

= 0.4

$$K2 = hf\left(x0 + \frac{h}{2},\ y0 + \frac{K_1}{2}\right)$$

$$= 0.2\left[\left(1 + \frac{0.2}{2}\right) \times \left(2 + \frac{0.4}{2}\right)\right]$$

$$= 0.2\left[\ 1 + 0.1\ \times\ 2 + 0.2\ \right]$$

$$= 0.2\left[\ 1.1\quad 2.2\ \right]$$

= 0.484

$K3 = hf(x0 + \frac{h}{2}, y0 + \frac{K_2}{2})$

$= 0.2\left[\left(1+\frac{0.2}{2}\right)x\left(2+\frac{0.484}{2}\right)\right]$

$= 0.49324$

$K4 = hf(x0 + h, y0 + K3)$

$= 0.2\left[1+0.2 \times 2+0.49324\right]$

$= 0.5983776$

$K = \frac{1}{6}(K1 + 2K2 + 2K3 + K4)$

$= \frac{1}{6}\ 0.4+2(0.484)+2(0.49324)+0.5983776$

$= 0.4921429$

$y1 = y0 + K$

$= 2 + 0.4921429$

$= 2.4921429$

(ii) x1 = x0 + h = 1 + 0.2 = 1.2, y1 = 2.4921429 and h = 0.2

$K1 = hf(x1, y1)$

$= 0.2[(1.2)\ (2.4921429)]$

$= 0.5981143$

$K2 = hf(x1 + \frac{h}{2}, y1 + \frac{K_1}{2})$

$= 0.2\left[\left(1.2+\frac{0.2}{2}\right)\times\left(2.4921+\frac{0.5981143}{2}\right)\right]$

$= 0.81824$

$$K3 = hf(x1 + \frac{h}{2}, y1 + \frac{K_2}{2})$$

$$= 0.2\left[\left(1.2+\frac{0.2}{2}\right)\times\left(2.4921+\frac{0.81824}{2}\right)\right]$$

$$= 0.7543283$$

$$K4 = hf(x0 + h, y0 + K3)$$

$$= 0.2\left[\ 1.2+0.2\ \times\ 2.4921+0.7543\ \right]$$

$$= 0.9090119$$

$$K = \frac{1}{6}(K1 + 2K2 + 2K3 + K4)$$

$$= 0.7753$$

$$y2 = y1 + K$$

$$= 2.4921 + 0.7753$$

$$= 3.26752$$

$$\therefore\ y(1.4) = 3.26752$$

□ □ □

Unit 5

Correlation & Regression

Q.1 Given a table of values for the function as :

X	0.1	0.2	0.3	0.4	0.5	0.6
y	5.1	5.3	5.6	5.7	5.9	6.1

Determine both the regression lines, and also prove that they intersect at $\left(\frac{\sum x}{n}, \frac{\sum y}{n}\right)$

Ans The values of $\sum x, \sum x^2, \sum y, \sum xy$ are computed as shown in the following table.

i	xi	yi	xi^2	xi^2	$xi\ yi$
1	0.1	5.1	0.01	26.01	0.51
2	0.2	5.3	0.04	28.09	1.06
3	0.3	5.6	0.09	31.36	1.68
4	0.4	5.7	0.16	32.49	2.28
5	0.5	5.9	0.28	34.81	2.95
6	0.6	6.1	0.36	37.21	3.66
n=6	$\sum x = 2.1$	$\sum y = 33.7$	$\sum y^2 = 0.91$	$\sum y^2 = 189.97$	$\sum xy = 12.14$

Regression line of y on x

Let the regression line of y on x be of type y on x be of y = $a_2 x + a_1$

$$a_2 = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2}$$

$$= \frac{\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2}$$

$$= \frac{33.7\ x\ 0.91 - 2.1\ x\ 12.14}{6\ x\ 0.91 - (2.1)^2}$$

=4.93

Thus, the regression line of y = x
Y = 1.97 x+ 4.93

Regression line of x on y

Let the regression line of x on y type
X = b_2 y +b_1

$$b_2 = \frac{n\sum xy - \sum y \sum x}{n\sum y^2 - (\sum y)^2}$$
$$= \frac{6\ x\ 12.14 - 33.7 x 2.1}{6.x189.97 - (33.7)^2}$$
= 0.50
$$b_1 = \frac{\sum x \sum y^2 - \sum y \sum xy}{n\sum y^2 - (\sum y)^2}$$
$$= \frac{2.1\ x\ 189.97 - 33.7\ x\ 12.14}{6\ x\ 189.97 - (33.7)^2}$$
= 2.46

Thus, the regression line of x on y is
$x = \ 0.50\ y - 2.46$

Now we would see that they intersect at $\left(\frac{\sum x}{n}, \frac{\sum y}{n}\right)$, $i.e.,$ (0.35, 5.62)

This we can be done if we can prove that both of these regression lines pass through it.

Regression liner of y on x, i.e.
Y = 1.97 x +4.63

Substituting x =0.35, we get
Y = 5.62

Which shows that the regression line of y on x passes through (0.35, 5.63)

Regression line of x on y, i.e.,

X = 0.50y – 2.46

Substituting y =5.62, we get
X = 0.50* 5.62 – 2.46
= 0.35

Which shows that the regression line of x on y also passes through (0.35, 5.62)
Since both the regression lines passes through (0.35, 5.62), thus we conclude that they intersect at the point $(\frac{\sum x}{n}, \frac{\sum y}{n})$

Q.2 Fit a second degree paradolic curve to the following data:

X	0	1	2	3	4
Y	0	1.8	1.3	2.5	6.3

Sol. Let the require curve have the equation
Y = a + bx + cx^2
$S^2 = \sum(a + bx + cx^2$
$S^2 = \sum(a + bx + cx)^2$
S^2 is minimum when a, b, c satisfied
the equation $\partial S^2 = 0, \partial S^2 = 0, \frac{\partial s^2}{\partial c} = 0$
Here n = 5

5a + b$\sum$x + c$\sum x^2$ = $\sum y$

a$\sum x$ + b$\sum x^2$ +C$\sum x^3$ = $\sum xy$

a$\sum x^2$ + b$\sum x^3$ +C$\sum x^4$ = $\sum x^2 y$

x	y	x^2	x^3	x^4	xy	x^2y
0	1	0	0	0	0	0
1	1.8	1	1	1	1.8	1.8
2	1.3	4	8	16	2.6	5.2
3	2.5	9	27	81	7.5	22.5
4	6.3	16	64	256	25.2	100.8
10	12.9	30	100	354	37.1	130.3

So,

5a + 10b + 30C = 12.9 -(5)
10a + 30b + 100C = 37.1 -(6)
30a + 100b + 354C = 130.3 -(7)

From (5) & (6)
10b + 40C = 11.3 - (8)

From (5) & (7)
40b + 174C = 52.9
(174-460) = 7.7
C = 0.55

Hence,

$10b = 11.3 - 40\left(\frac{55}{100}\right)$
= -1.07
Hence, 5a = 12.9 – 10.7 – 16.5
= 7.1
Hence the equation to the required parabola is
$Y = 1.42 - 1.07x + 0.55x^2$

Q.3 Find the best fit of line from following data.

U	**-5**	**-4**	**-3**	**-2**	**-1**	**0**	**1**	**2**	**3**	**4**
Y	**45**	**52**	**54**	**63**	**62**	**68**	**75**	**76**	**92**	**88**

Sol. Let the line of best fit be
Y = a + bu
The normal equation are
$\sum y = 10a + b\sum u$
$\sum uy = a \sum u + b\sum u^2$

u	y	x^2	uy
-5	45	25	-225
-4	52	16	-208

-3	54	9	-162
-2	93	4	-126
-1	62	1	-62
0	68	0	0
1	75	1	75
2	76	2	152
3	92	3	276
4	88	4	352
$\sum u = -5$	$\sum u = 675$	$\sum u^2 = 85$	$\sum uy = 72$

The equations will be
675 = 10 a- 5b
72 = - 5a + 85b
= a = 6.998
B = 4.96
So the line of best fit is
Y= 6.998 + 4.96u

Q.4 **The marks secured by students in Mathematics & Statistics are given below:**

Roll No.	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Maths	**10**	**15**	**12**	**17**	**13**	**16**	**25**	**14**	**22**
Statics	**30**	**42**	**45**	**46**	**33**	**34**	**40**	**35**	**39**

Calculate the rank correlation coefficient.

Sol.

Roll No.	1	2	3	4	5	6	7	8	9
Maths	9	5	8	3	7	4	1	6	2
Statics	9	3	2	1	8	7	4	6	5
Difference	0	2	6	2	-1	-3	-3	0	-3
d^2	19	12	19	10	19	14	9	20	13